UNIVERSITY OF HAWAII
LIBRARY

ADVANCES IN PHYSICS

A QUARTERLY SUPPLEMENT
of the
PHILOSOPHICAL MAGAZINE

EDITOR
PROFESSOR B. H. FLOWERS, M.A., D.Sc., F.R.S.

CONSULTANT EDITOR
PROFESSOR N. F. MOTT, M.A., D.Sc., F.R.S.

EDITORIAL BOARD
SIR LAWRENCE BRAGG, O.B.E., M.C., M.A., D.Sc., F.R.S.
SIR GEORGE THOMSON, M.A., D.Sc., F.R.S.
PROFESSOR A. M. TYNDALL, C.B.E., D.Sc., F.R.S.

VOLUME 10 JULY 1961 NUMBER 39

PRICE per part 25*s.* plus postage
PRICE per annum £4 15*s.* 0*d.* post free

PRINTED AND PUBLISHED BY TAYLOR & FRANCIS LTD
RED LION COURT, FLEET ST., LONDON, E.C.4

REVISED CHEAPER EDITION FOR LIBRARIES AND SCHOOLS

A History of Mathematics

From antiquity to the early nineteenth century

By J. F. SCOTT, B.A., D.Sc., Ph.D.

Vice-Principal of St. Mary's College, Strawberry Hill, Twickenham, Middlesex

Author of *The Scientific Work of René Descartes* (1596–1650),
Mathematical Work of John Wallis, D.D., F.R.S. (1616–1703), and other works

CONTENTS: Mathematics in Antiquity—Greek Mathematics—The Invention of Trigonometry—Decline of Alexandrian Science and the Revival in Europe—Mathematics in the Orient—Progress of Mathematics during the Renaissance—New Methods in Geometry—The Rise of Mechanics—The Invention of Decimal Fractions and of Logarithms—Newton and the Calculus—Taylor and Maclaurin, the Bernoullis and Euler, Related Advances—The Calculus of Variations, Probability, Projective Geometry, Non-Euclidean Geometry—Theory of Numbers—Lagrange, Legendre, Laplace, Gauss. This volume is intended primarily to help students who desire to have a knowledge of the development of the subject but who have too little leisure to consult works and documents. The author has availed himself of the facilities afforded by the Royal Society and other learned Societies to reproduce extracts from manuscripts and many scarce works.

Size $9\frac{3}{4}'' \times 6\frac{3}{4}''$. 266 pp. **Price 27s. 6d.** plus postage and packing 2*s.* 0*d.*

Some Reviews of the First Edition

" The invention of trigonometry, decimal fractions, logarithms and the calculus are each discussed clearly and concisely. The book is easy to read for anybody who knows the elements of mathematics and, although not free from minor errors, can be strongly recommended."—*British Book News*, April 1958.

" Physicists will find that the development in applied mathematics are clearly set out, from ancient times, through that of Archimedes, to the mechanics of the sixteenth century when interest was revived. Significant advances made by Stevin, Galileo, Descartes, Huygens and others are stressed, and help the reader to appreciate what Newton achieved. There are useful appendices giving brief biographical notes on mathematical topics and terminology, followed by a bibliography."—*Proceedings of The Physical Society*, September 1958.

" . . . it has been written with clarity and balance, and the excellent printing helps to make it a pleasure to read."—*The Times Educational Supplement*, 21 March 1958.

" . . . his (Dr. Scott's) wide knowledge of the material, his careful description of methods combine to provide an account which at times gives a sense of the excitement of discovery."—*Nature*, 26 July 1958.

" The printers and publishers are to be congratulated upon having produced such an attractive volume, ... We feel sure the book will be received with delight by all those interested in mathematical histories."—*BEAMA Journal*, August 1958.

" The work comes to life mainly because of his admirable use of the writings of mathematicians themselves, which vividly illustrates the great difficulties under which many of them laboured. This is not a book for the layman but both the student and anyone to whom figures are a fascination, will find the subject clearly and pleasantly presented."—*Technical Bookguide*, March 1958.

Printed and Published by

TAYLOR & FRANCIS LTD

RED LION COURT, FLEET STREET, LONDON, E.C.4

CONTENTS

The Theory of Nuclear Matter. By J. S. Bell, CERN, Geneva and E. J. Squires, Department of Applied Mathematics and Theoretical Physics, Cambridge 211

The Theory of Nuclear Matter

By J. S. Bell
CERN, Geneva

and E. J. Squires
Department of Applied Mathematics and Theoretical Physics, Cambridge

CONTENTS

	PAGE
§ 1. NUCLEAR MATTER.	212
1.1. Properties of Nuclear Matter.	212
1.2. The Fermi Gas.	214
1.3. Interactions.	215
1.4. Perturbation Theory.	218
1.5. The Hard Core in the Many-body Problem.	222
§ 2. TRIAL WAVEFUNCTION METHODS.	224
2.1. Two-body Correlations and the Cluster Expansion.	224
2.2. The Euler Equations for the Correlation Function.	227
2.3. Numerical Calculations.	229
2.4. The Effective Potential.	232
2.5. The Bethe–Goldstone Equation.	234
§ 3. BRUECKNER THEORY.	237
3.1. Introduction.	237
3.2. Perturbation Theory for Many-body Systems.	238
3.3. The Linked Diagram Series.	239
3.4. The Low Density Limit.	247
3.5. The K-matrix Equation.	249
3.6. Third-order Corrections.	256
3.7. The Self-consistent Field.	262
3.8. The Brueckner–Gammel Calculation.	269
3.9. Separation and Rearrangement Energies.	272
3.10. The Optical Model Potential.	275
3.11. Final Remarks.	280
§ 4. SINGULARITIES AND SUPERFLUIDITY.	280
4.1. Introduction.	280
4.2. The K-matrix Singularity.	281
4.3. The Inclusion of Additional Terms in the Perturbation Series.	284
4.4. Ladder Diagrams at Finite Temperatures.	290
4.5. The Bardeen–Cooper–Schrieffer Theory.	293
4.6. BCS Theory at Finite Temperatures.	300
4.7. The Effect of Residual Terms in the Hamiltonian.	301
4.8. The Superfluidity of Nuclear Matter.	302
§ 5. CONCLUSION.	308

ABSTRACT

This article is an account of the present position of theories of nuclear matter initiated some years ago mainly by Jastrow and Brueckner. Section 1 introduces the basic ideas. Section 2 discusses the variational approach. Section 3 develops methods involving partial summations in perturbation theory. The difficulties confronting the latter approach, and their partial resolution by ideas from superconductivity theory, are analysed in § 4.

Much of the formalism is relevant also in the theory of individual nuclei (see for example the review by Eden 1959) and in fields other than nuclear physics (see for example *The Many Body Problem*, Les Houches, 1958). However, in the selection of material our criterion has been relevance for the ground state of nuclear matter.

§ 1. NUCLEAR MATTER

1.1. *Properties of Nuclear Matter*

IT has long been known that nuclei, unlike atoms, exhibit fairly well-defined radii and moreover that the nuclear volume is proportional to the number of nucleons. This led to the idea that, for some purposes, nuclei could be regarded as droplets of substantially incompressible 'nuclear matter'. If the coulomb forces, which prevent the occurrence of arbitrarily large nuclei in nature, are switched off, it becomes possible to contemplate in theory the existence of indefinitely large portions of this matter. The resulting spatial homogeneity greatly simplifies the theory. The properties which such a theory should try to account for must of course be obtained by extrapolation from finite nuclei.

The normal density of nuclear matter is obtained most directly from the electron scattering experiments (reviewed by Elton 1961 and Ravenhall 1958). These are consistent with the assumption that all nuclei have the same density in a central unform region of varying extent, and, surrounding this, have a surface region of constant thickness ($\approx 2f$). When so analysed (Elton 1958) the experiments yield for the central regions a volume per nucleon $\frac{4}{3}\pi r_0^3$ with

$$r_0 = 1{\cdot}12f. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.1)$$

We refer sometimes to the 'mean spacing' of nucleons, $1{\cdot}6\, r_0$, which is defined as the cube root of the volume per nucleon.

The energy density of nuclear matter can be inferred from the Weizsacker semi-empirical mass formula (reviewed by Green 1958). This expresses the energy per nucleon of a finite system as the value appropriate in nuclear matter, plus corrections for finite size (surface energy), electrical forces (coulomb energy) and departure of the proton–neutron ratio from the value unity which we attribute to standard nuclear matter (symmetry energy). Simple estimates of each effect result in the formula

$$(E/A) = -a_1 + a_2 A^{-1/3} + a_3 Z^2 A^{-4/3} + a_4 (N-Z)^2 A^{-2} \quad . \quad . \quad (1.2)$$

for the energy E of a nucleus with A nucleons, N neutrons and Z protons. Fitting the empirical data gives, according to Green,

$$\left.\begin{aligned} a_1 &= 15{\cdot}8\,\text{Mev}, \\ a_2 &= 18{\cdot}0\,\text{Mev}, \\ a_3 &= 0{\cdot}72\,\text{Mev}, \\ a_4 &= 24\,\text{Mev}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.3)$$

To account for the quantity a_1 is clearly one of the main aims of a theory of nuclear matter. To predict a_2 would require a discussion of finite nuclei, or at least of the semi-infinite system. The quantity a_3 in the coulomb energy is readily interpreted, in quantitative agreement with the electron scattering data, in terms of nuclear size. Finally a_4 is again a proprety of the infinite system, to be obtained by varying therein the neutron–proton ratio.

However, the treatment of symmetry energy in (1.2) is probably oversimplified. Cameron (1957) has proposed that the final term be replaced by

$$\left(\frac{N-Z}{A}\right)^2 (a_{SV} - a_{SS} A^{-1/3}) \qquad (1.4)$$

which includes a surface as well as a volume effect. According to Green the data then require

$$\left.\begin{aligned} a_{SV} &\approx 30\,\text{Mev}, \\ a_{SS} &\approx 40\,\text{Mev}. \end{aligned}\right\} \qquad (1.5)$$

Only a_{SV} is relevant for nuclear matter.

The analysis of electron scattering referred to above, and the Weizsacker formula, involve the idea of nuclear incompressibility. This of course is an idealization. A measure of the compressibility is given by the parameter

$$K = r_0{}^2 \frac{d^2}{dr_0{}^2}(E/A) \qquad (1.6)$$

(where E/A is the energy per particle in nuclear matter and r_0 defines the density) evaluated at the value of r_0 for which

$$\frac{d}{dr_0}(E/A) = 0 \qquad (1.7)$$

(this just implies zero pressure, i.e. the absence of stress). Now the central regions of nuclei are not in fact quite free from stress; on the simplest picture the matter there is compressed by surface tension and distended by coulomb forces. On a naive theory this results in a change of mean density given by

$$\frac{\delta r_0}{r_0} = \tfrac{3}{5}\frac{Z^2e^2}{K r_0 A^{4/3}} - \frac{2a_2}{K A^{1/3}} \qquad (1.8)$$

away from the normal value defined by (1.7). For actual nuclei the second term is always the larger and δr_0 is negative. With $K = 170$ Mev (the theoretical figure of Brueckner and Gammel 1958) $(\delta r_0/r_0)$ varies from about 0·02 in ^{107}Ag to about 0·01 in ^{197}Au. This change would not be observable, but much smaller values of K would be ruled out by the electron scattering data. The rather smaller value

$$K \approx 70\,\text{Mev} \qquad (1.9)$$

was indeed obtained at one time from the 'isotope shift' in atomic spectra (Ford and Hill 1955) which can be attributed in part to the variation in mean density predicted by (1.8) for nuclei of equal Z and different A. However, Bodmer (1958) has observed that the unknown variation of symmetry energy with density is also important, and prevents the unambiguous derivation of a value for K. Other refinements of the theory are also important here (Breit 1958). Further estimates of compressibility are obtained from semi-empirical theories of the nuclear surface energy and shape (Wilets 1958, Skyrme 1956) where K enters as an important parameter. Comparing the theory with various versions of the Weizsacker formula, Wilets obtains values

$$K \approx 175,\ 218,\ 302\,\text{Mev}. \qquad (1.10)$$

The compressibility is therefore very poorly known.

Once finite compressibility is admitted the possibility must be considered that the central regions of heavy nuclei are not in fact of uniform density, for coulomb forces will tend to produce a central depression. This is not decided empirically, for electron scattering is quite insensitive to central density (Ravenhall 1958, Elton 1961). In a very naive picture which ignores possible neutron–proton separation and forces due to density gradients, one obtains a coulomb effect at the centre of 2·5 times the mean value given by the first term in (1.8). With $K \approx 170$ Mev this implies a central depression of density by some 20% in heavy nuclei. However, in the more elaborate theory of Wilets (1958) the effect is almost entirely suppressed, for reasons which are obscure.

1.2. *The Fermi Gas*

The simplest model of nuclear matter, and the point of departure of nearly all current theories†, is the ideal fermi gas. This gives already first crude estimates of some of the quantities mentioned above. Consider a cubical box of volume Ω, which is finally made very large, on the boundaries of which wave functions satisfy the conventional periodic boundary conditions. It is assumed that for sufficiently large Ω the precise boundary conditions are unimportant‡, and the periodicity condition is convenient because it leaves the theory accurately translationally invariant. The permitted states of individual particles are plane waves with momentum vectors

$$(k_x, k_y, k_z) = 2\pi(l, m, n)\Omega^{-1/3}$$

where l, m, n are positive or negative integers. The ground state is a determinantal wave function in which all plane waves with $|\mathbf{k}|$ less than or equal to the fermi momentum k_f are occupied, each with four particles—corresponding to the four states of spin (including isobaric spin). It is useful to denote occupied states by symbols $\mathbf{m}, \mathbf{m}'\ldots$, unoccupied states by $\mathbf{l}, \mathbf{l}'\ldots$, and states in general by $\mathbf{k}, \mathbf{k}'\ldots$. Moreover it is usually convenient to suppose that these symbols specify spins as well as momenta, to avoid encumbering formulae with special spin indices.

---

† See however Trainor *et al.* (1958) and Trainor and Sood (1959).

‡ It would be desirable to start with a theory of large finite nuclei and demonstrate that the periodic boundary condition model correctly represents their central regions. This has not been done. Some preliminary observations are made by Eden (1958).

The fermi momentum is related to the density parameter r_0 through the mean volume per nucleon

$$\tfrac{4}{3}\pi r_0^3 = \Omega\left(\sum_{\mathbf{m}} 1\right)^{-1}.$$

For large Ω there are

$$\Omega\, d\mathbf{m}\,(2\pi)^{-3} \qquad (1.11)$$

states available in a region $d\mathbf{m}$ of momentum space, where

$$d\mathbf{m} = dm_x\, dm_y\, dm_z.$$

Then, with a factor 4 from spin summation,

$$\tfrac{4}{3}\pi r_0^3 = \left\{4\int \frac{d\mathbf{m}}{(2\pi)^3}\right\}^{-1}$$

or

$$k_f r_0 = (9\pi/8)^{1/3} = 1{\cdot}52. \qquad (1.12)$$

With $r_0 = 1{\cdot}12f$,

$$k_f = 1{\cdot}35 f^{-1}. \qquad (1.13)$$

The corresponding kinetic energy is

$$k_f^2/2M = 38\,\text{Mev}, \qquad (1.14)$$

and the mean kinetic energy per particle is

$$\tfrac{3}{5}\, k_f^2/2M = 23\,\text{Mev}. \qquad (1.15)$$

In the absence of interactions the energy per particle does not of course exhibit a minimum at any value of r_0. But with r_0 fixed at the desired value, values can be obtained for compressibility and symmetry energy. The former is

$$K = r_0^2(d^2/dr_0^2)(3k_f^2/10M) = (18k_f^2/10M)$$

$$\approx 138\,\text{Mev}. \qquad (1.16)$$

The symmetry energy, obtained by considering slightly different fermi momenta for neutrons and protons subject to fixed total density, is

$$a_{\text{sv}} = (k_f^2/6M) \approx 13\,\text{Mev}. \qquad (1.17)$$

These are already of the right order of magnitude, but comparable contributions arise from internucleon forces.

1.3. *Interactions*

The most fundamental procedure at this stage would be to let the fermi gas interact with a meson field and to evaluate the resultant energy shift. Such a programme is hardly practical, in view of the very limited success of meson theory even in the two-body problem. Nevertheless some speculative attempts have been made, based on the extreme simplification that the discrete nature of the nucleons as sources of the meson field is ignored and that they be replaced by an averaged density distribution

(Skyrme 1955, Johnson and Teller 1955, Schiff 1952, 1954). There results a collective meson field, and individual nucleons interact with this rather than with one another. This picture of bulk nuclear matter is not incompatible with the action of residual two-body forces between particles outside closed shells, as required in the shell model when it goes beyond the extreme single particle version. Nor is it incompatible with strong two-body correlations in the lower density surface region of a nucleus, which might possibly be enough to account for various high energy phenomena which have been put forward as establishing the existence of such correlations in nuclear matter (Brueckner *et al.* 1955). However, while such theories are not ruled out by experiment, and certainly not by meson theory in its present state, they are perhaps rather implausible. The mean spacing of particles in nuclear matter is slightly greater than the pion compton wavelength; thus, while the meson clouds of several nucleons will often overlap. it seems unlikely that they fuse into a collective whole. Whatever the case may be, these theories have developed little recently, and we do not consider them here.

The more traditional approach has been to suppose the meson field formally eliminated, leaving an effective hamiltonian involving nucleons only, and to determine the latter as well as possible by experiment. There will be terms involving the coordinates of two, three, four, . . . and arbitrarily many nucleons. Much can be learned about the two-body interaction from nucleon–nucleon scattering experiments and deuteron properties. Something could then, in principle, be learned about three-body forces from ^{3}He and ^{3}H and from nucleon–deuteron scattering, but this would require rather well-determined two-body forces and a reliable solution of the wave-mechanical three-body problem (see for example Blatt and Derrick 1960)†. In fact it seems impossible to say more about the experimental existence of three-body forces, and of many-body forces in general, than that so far no clear evidence for them has been recognized. Guidance from meson theory is equally indecisive. Drell and Huang (1953), where references to earlier literature are given, considered the problem in lowest order perturbation theory with the relativistic PS–PS meson theory. They found the three-body force to be of very great importance in nuclear matter. However their leading terms involved s-wave meson–nucleon interaction, for which perturbation theory is now known to be grossly wrong. In this respect the work of Fujita and Miyazawa (1957) is more realistic‡. They use no-recoil meson theory and by dispersion relations relate the simplest contribution to the three-body force to observed meson–nucleon scattering. The expectation value of this force in the ideal fermi gas at normal density is found to be only 0·2 MeV. As the authors say, the result is not to be taken too seriously. The theory is reliable for at most the long-range part of the force and probably, as for the two-body force, quite inadequate for the bulk of it. The calculation does however provide some excuse for the customary position of leaving till later the consideration of many-body forces in nuclear theory. This position we adopt.

---

† These authors now find that the Brueckner–Gammel potential of this section, unsupplemented by any three-body force, is in definite disagreement with empirical properties of the three-nucleon system (Derrick *et al.* 1961).

‡ See also Brueckner *et al.* (1954) and Smith and Sharp (1960). The latter authors abtain a contribution to A/E of just over 1 MeV at normal density.

It remains to make some brief remarks about the two-body potential (reviewed by Phillips 1959, Gammel and Thaler 1960 and MacGregor *et al.* 1960). This is not determined observationally, even in principle, but involves an element of convention (see Phillips 1959 and the following paragraph). It is usually taken to be velocity independent, or nearly so, and simple assumptions made about its shape. Such a potential can still contain numerous parameters; the present data require the introduction of many, without being able to fix their values very definitely. However, some features appear to be clear. At least in the singlet s-state the interaction appears to be strongly repulsive at short distances. This is usually represented by making the potential infinity positive within a radius of about 0·4*f*—the so-called 'hard core'. Outside this there is an attractive central potential mainly in even states (a Serber force). The quadrupole moment of the deuteron requires a tensor force, and in order to fit both high and low energy scattering it is probably necessary to add an interaction between spin and orbital angular momentum (this is a departure from velocity independence). In nuclear matter the tensor and spin orbit forces do not contribute to the energy in first order, and are often omitted in preliminary studies. So far only Brueckner and Gammel (1958) have used a potential with all the features listed†. This they obtained by substantial modification of one due to Gammel and Thaler (1957) which did not fit low energy data (the effective range region). Their potential is a hard core (in all states) of radius 0·4*f* outside of which

$$\begin{aligned} v = (\tfrac{1}{2} - \tfrac{1}{2}P^{\sigma}P^{\tau})\{ & -877{\cdot}4\,(\tfrac{1}{2} + \tfrac{1}{2}P^{\sigma})\exp(-2{\cdot}091r)/2{\cdot}091r \\ & -434\quad(\tfrac{1}{2} - \tfrac{1}{2}P^{\sigma})\exp(-1{\cdot}45r)/1{\cdot}45r \\ & -159{\cdot}4\,S\exp(-1{\cdot}045r)/1{\cdot}045r \\ & -5000\,S'\exp(-3{\cdot}7r)\}/3{\cdot}7r \\ +(\tfrac{1}{2} + \tfrac{1}{2}P^{\sigma}P^{\tau})\{ & -14{\cdot}0\,(\tfrac{1}{2} + \tfrac{1}{2}P^{\sigma})\exp(-r)/r \\ & +130\quad(\tfrac{1}{2} - \tfrac{1}{2}P^{\sigma})\exp(-r)/r \\ & +22\,S\exp(-0{\cdot}8r)/0{\cdot}8r \\ & -7315\,S'\exp(-3{\cdot}7r)\}/3{\cdot}7r \end{aligned}$$

in MeV, with r in fermis, where

$$S = 3\frac{\boldsymbol{\sigma}_1\,.\,\mathbf{r}\,\boldsymbol{\sigma}_2\,.\,\mathbf{r}}{r^2} - \boldsymbol{\sigma}_1\,.\,\boldsymbol{\sigma}_2,\quad S' = \tfrac{1}{2}(\mathbf{r}_1 - \mathbf{r}_2)\times(\mathbf{p}_1 - \mathbf{p}_2).$$

The precision with which the parameters are fixed by experiment is not stated. However the weak triplet-odd interaction in particular (the 5th term above) is fixed quite poorly. It is heavily weighted in the nuclear matter calculation, and the quoted value was chosen to obtain there the desired energy per particle, which would otherwise be uncertain by several Mev (J. Gammel, private communication). The Brueckner–Gammel potential does not at long range agree with the one-pion exchange potential of meson theory. This asymptotic part of the mesonic potential is now thought to be both theoretically reliable and experimentally supported (MacGregor *et al.* 1960). It is to be hoped that detailed potentials with the appropriate long-range character will be forthcoming and the nuclear matter calculations repeated.

---

† Also Scott and Moszkowski (see § 3.5).

The hard core, as will be seen, is a considerable complication in the many-body problem. It has occasionally been suggested therefore that some velocity dependent (or non-local) potential be used instead (see, for example, Peierls 1960), which could equally well become repulsive at high energy. The empirical success of the hard core does not exclude this possibility, for from any potential which fits the data another is generated by any canonical transformation which leaves unaltered the asymptotic wave function. It should be noted however that such a transformation alters the interpretation of the wavefunction and in general changes the expressions for quantities like, for example, the deuteron quadrupole moment. Moreover, potentials equivalent in the two-body system are not immediately so in the many-body case. This is compensated by the fact that the transformation, generalized to the many-body system, alters also the many-body forces or introduces some if they were absent. The three-body force of Fujita and Miyazawa, for example, is complementary only to the static two-body force (inevitable in static meson theory) explicity assumed in their work. However, these matters have received little attention. The only non-static potential we will refer to is the so-called 'separable' potential, and that less for its ability to fit data (Yamaguchi 1954) than for its great simplicity in formal investigations like those of § 4 where realism is a secondary consideration.

1.4. *Perturbation Theory*

Suppose for the moment that the nucleon interaction is a non-singular two-body force of the type for long conventional in nuclear physics. A non-singular potential is here one which has a finite fourier transform; this does not exclude for example the Yukawa shape. Such a potential can fit data in the region of effective range theory, and some features of higher energy scattering can be obtained with the Serber exchange mixture. Perturbation theory can be used to estimate the resultant energy shift in nuclear matter away from that for the ideal gas. This was carried to second order by Euler (1937) for a gaussian potential, by Huby (1949) for a Yukawa, and by Thouless (1957) for an exponential well acting in *S*-states only; their general results are similar. It might be feared that the series would converge slowly if at all, for nuclear forces are not weak. In nuclear matter particles collide with relative velocities up to $(2k_f/M)$, corresponding to incident energies in scattering experiments up to

$$\frac{M}{2}\left(\frac{2k_f}{M}\right)^2 = 4\,\frac{k_f^2}{2M} \approx 150\ \text{Mev.} \qquad (1.18)$$

Born approximation in collision theory is poor over all of this range (Jost and Pais 1951). Despite this the first approximation to the energy shift in nuclear matter, as judged by the size of the next term, is reasonable (Swiatecki 1956, Bethe 1956). Thus the perturbation results, and it is their main interest, support the idea of Weisskopf (1950) that the ideal gas is a much better starting point than might have been expected because the exclusion principle suppresses much wave function admixture that could otherwise occur.

In first order the energy shift is the expectation value of the potential in the unperturbed state; i.e.

$$(\Delta E)_1 = \tfrac{1}{2}\sum_{\mathbf{m},\,\mathbf{m}'}\{\langle \mathbf{m}\mathbf{m}'\,|v|\,\mathbf{m}\mathbf{m}'\rangle - \langle \mathbf{m}\mathbf{m}'\,|v|\,\mathbf{m}'\mathbf{m}\rangle\} \qquad (1.19)$$

in terms of matrix elements of the interaction, v, between pairs of single particle states normalized in the box. The presence of the exchange term will often be indicated by the notation

$$(\Delta E)_1 = \tfrac{1}{2}\sum_{\mathbf{m},\,\mathbf{m}}\langle \mathbf{m}\mathbf{m}'|v|\mathbf{m}\mathbf{m}' - \mathbf{m}'\mathbf{m}\rangle. \qquad (1.20)$$

In calculating spin sums the form

$$(\Delta E)_1 = \tfrac{1}{2}\sum_{\mathbf{m},\,\mathbf{m}'}\langle \mathbf{m}\mathbf{m}'\,|v(1 - P^r P^\sigma P^\tau)|\,\mathbf{m}\mathbf{m}'\rangle \qquad (1.21)$$

is useful, where (P^r, P^σ, P^τ) are two-body exchange operators in space, spin and isobaric spin. Consider for simplicity a mixture of only Wigner and Majorana forces

$$\left.\begin{aligned} v &= (W - MP^\sigma P^\tau)u, \\ W + M &= 1, \end{aligned}\right\} \qquad (1.22)$$

where u depends only on interparticle distance. The spin sum implied in (21) gives a factor which is the trace over the spin states of two particles

$$\text{trace}\,(W - MP^\sigma P^\tau)(1 - P^\sigma P^\tau P^r) = 16\{(W - \tfrac{1}{4}M) + (M - \tfrac{1}{4}W)P^r\}. \qquad (1.23)$$

The matrix element $\langle \mathbf{k}_1\mathbf{k}_2\,|v|\,\mathbf{k}_1'\mathbf{k}_2'\rangle$ has the spin independent factor

$$\Omega^{-2}\int d\mathbf{r}_1\,d\mathbf{r}_2\,u(\mathbf{r}_1 - \mathbf{r}_2)\exp\{i(\mathbf{k}_1' - \mathbf{k}_1)\,.\,\mathbf{r}_1 + i(\mathbf{k}_2' - \mathbf{k}_2)\mathbf{r}_2\}$$
$$= \Omega^{-2}\,\delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_1' - \mathbf{k}_2')\,\tilde{u}\,(\mathbf{k}_1 - \mathbf{k}_1'), \qquad (1.24)$$

where

$$\tilde{u}(\mathbf{k}) = \int d\mathbf{r}\,u\,(r)\exp(-i\mathbf{k}\,.\,\mathbf{r}). \qquad (1.25)$$

Converting momentum sums into integrals by (1.11), and with

$$A = 4\Omega\,\frac{d\mathbf{m}}{(2\pi)^3} \qquad (1.26)$$

we obtain for the energy shift per particle

$$\left(\frac{\Delta E}{A}\right)_1 = 2\int \frac{d\mathbf{m}}{(2\pi)^3}\frac{d\mathbf{m}'}{(2\pi)^3}\{(W-\tfrac{1}{4}M)\,\tilde{u}(0)+(M-\tfrac{1}{4}W)\,\tilde{u}(\mathbf{m}-\mathbf{m}')\}\Big/\int\frac{d\mathbf{m}}{(2\pi)^3}. \quad . \quad . \quad . \quad (1.27)$$

At high density (large k_f) the exchange term, involving $\tilde{u}(\mathbf{m}-\mathbf{m}')$, becomes relatively negligible because of the fall off of $\tilde{u}(\mathbf{k})$ for large $\mathbf{k}$. The direct term, involving $u(0)$, is simply proportional to k_f^3, and at high density dominates also the kinetic energy ($\propto k_f^2$). Since $\tilde{u}(0)$ is negative for forces attractive in S-states, it is then possible by going to arbitrarily high density to obtain an indefinitely low energy per particle, and the system is unstable against collapse, unless

$$(W-\tfrac{1}{4}M)<0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.28)$$

This condition is quite accurately necessary, because the result of first order perturbation theory (not of higher orders) has a variational significance; the expression for the total energy is just the expectation value of the total hamiltonian (in the unperturbed state) and the true energy is lower. It can be shown to be sufficient as well as necessary (Wigner 1936). For a general account of such 'saturation conditions' see Blatt and Weisskopf (1952). The Serber force ($W=M=\tfrac{1}{2}$) does not satisfy (1.28), and nuclear saturation is now attributed only in part to the exchange component in the force, which is supplemented by the hard core.

To obtain a numerical value for $(\Delta E/A)_1$ at arbitrarily prescribed density we will use a Yukawa potential

$$u=V\frac{\exp(-\mu r)}{\mu r}, \quad \tilde{u}(k)=\frac{4\pi V}{\mu^3+\mu k^2}. \quad . \quad . \quad . \quad . \quad . \quad (1.29)$$

Suitable parameters (used for example by Bethe 1956 and Brueckner 1958) are

$$V=49\,\text{Mev}, \quad \mu^{-1}=1{\cdot}19f. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.30)$$

These fit low energy singlet-state scattering. Performing the integrals in (1.27) (Huby 1949) we get

$$\left.\begin{aligned}\left(\frac{\Delta E}{A}\right)_1 &= \frac{3V}{2\pi}\left\{(W-\tfrac{1}{4}M)\Omega_0\left(\frac{k_f}{\mu}\right)+(M-\tfrac{1}{4}W)\Omega_1\left(\frac{k_f}{\mu}\right)\right.,\\ \text{with}\qquad \Omega_0(a) &= \frac{8a^3}{9},\end{aligned}\right\} \quad (1.31)$$

$$\Omega_1(a)=\left(2a-\frac{1}{3a}\right)+\left(\frac{1}{a}+\frac{1}{12a^3}\right)\log(1+4a^2)-\tfrac{8}{3}\tan^{-1}2a.$$

For $r_0=1{\cdot}12f$ and $(k_f/\mu)=1{\cdot}61$, $\Omega_0\approx 3{\cdot}7$ and $\Omega_1\approx 1{\cdot}2$.

The Serber mixture then gives

$$(\Delta E/A)_1\approx -43\,\text{Mev}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.32)$$

With the kinetic energy 23 MeV per particle, the binding energy per particle is 20 MeV. For the just saturating mixture $W=\frac{1}{5}$, $M=\frac{4}{5}$, we would have obtained -2 MeV. The experimental value is 16 MeV.

Consider now the second order energy shift. This results from a two stage process; first a pair of particles is excited into states which, because of the exclusion principle, may be taken to be above the Fermi surface; these particles must then be returned to their original states, possibly interchanged. Thus

$$(\Delta E)_2 = \tfrac{1}{2} \sum_{\mathbf{m},\mathbf{m}',\mathbf{l},\mathbf{l}'} \frac{\langle \mathbf{mm}' | v | \mathbf{ll}' \rangle \langle \mathbf{ll}' | v | \mathbf{mm}' - \mathbf{m}'\mathbf{m} \rangle}{(m^2 + m'^2 - l^2 - l'^2)/2M}. \quad . \quad . \quad (1.33)$$

The sum is over all spin states; $| ll' \rangle \langle ll' |$ is just the unit operator in spin space, so the required spin summation is

$$\begin{aligned} \tfrac{1}{16} \text{ trace} (W - MP^\sigma P^\tau)^2 (1 - P^\sigma P^\tau P^r) \\ &= (W^2 + M^2 - \tfrac{1}{2}WM) + (2WM - \tfrac{1}{4}W^2 - \tfrac{1}{2}M^2)P^r. \quad . \quad . \quad (1.34) \\ &= \alpha + \beta P^r. \end{aligned}$$

Using momentum conservation, implied by the δ-function in (1.24), we introduce

$$\left.\begin{aligned} \mathbf{k} &= \mathbf{l} - \mathbf{m} = \mathbf{m}' - \mathbf{l}', \\ \mathbf{k}' &= \mathbf{l} - \mathbf{m}' = \mathbf{m} - \mathbf{l}' = \mathbf{k} + \mathbf{m} - \mathbf{m}'. \end{aligned}\right\} \quad . \quad . \quad . \quad (1.35)$$

Then, converting sums to integrals,

$$\left(\frac{\Delta E}{A}\right)_2 = -4M \int\limits_{\substack{|\mathbf{m}+\mathbf{k}| > k_f \\ |\mathbf{m}'-\mathbf{k}| > k_f}} \frac{d\mathbf{m}}{(2\pi)^3} \frac{d\mathbf{m}'}{(2\pi)^3} \frac{d\mathbf{k}}{(2\pi)^3} \frac{\alpha \tilde{u}(k)\tilde{u}(k) + \beta \tilde{u}(k)\tilde{u}(k')}{2\mathbf{k}\,.\,\mathbf{k}'} \Bigg/ \int \frac{d\mathbf{m}}{(2\pi)^3} \quad . \quad . \quad . \quad (1.36)$$

where, as always, $\mathbf{m}$ $\mathbf{m}'$ integrations run only below the fermi surface. For the Yukawa force

$$\left(\frac{\Delta E}{A}\right)_2 = -\frac{3V^2M}{2\pi^2\mu^2} \left\{ \alpha\Omega_2\left(\frac{k_f}{\mu}\right) + \beta\Omega_3\left(\frac{k_f}{\mu}\right) \right\}. \quad . \quad . \quad (1.37)$$

The integral Ω_2 has been reduced by Huby to a lengthy combination of one dimensional integrals involving elementary functions. We refer to his equation (26) for the expression; his diagram includes a plot of $(0{\cdot}52\Omega_2)$ against μr_0 (r_0 in his notation). The exchange integral Ω_3 is more difficult but can be bounded by observing that $(\Delta E)_2$ must be negative for arbitrary exchange mixture (because the first order shift is an upper bound) and that $\tilde{u}(k)$ (1.29) has everywhere the same sign; thus

$$0 < \Omega_3 < \Omega_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.38)$$

For the above parameters and at normal density $\Omega_2 \approx 0{\cdot}50$. Then, with the Serber mixture,

$$4{\cdot}6 \text{ MeV} > (\Delta E/A)_2 > 2{\cdot}3 \text{ MeV}. \quad . \quad . \quad . \quad . \quad (1.39)$$

This is only between 5% and 10% of the first-order contribution. At high density Ω_2 increases only logarithmically with k_f, while Ω_0 goes as k_f^3, so the convergence improves. To some extent this is to be expected simply because the two-body collisions are increasingly energetic. However, the exclusion principle plays a large part. To see this one can drop the restriction on the k integration in (1.36), taking the principal part of the resulting singular integral; Ω_2 is replaced by

$$(\Omega_2'a) = \frac{\pi a}{3}\left\{\left(3-\frac{1}{2a^2}\right)+\left(\frac{3}{2a^2}+\frac{1}{8a^4}\right)\log\left(1+4a^2\right)-\frac{4}{a}\tan^{-1}2a\right\}. \quad . \; . \; . \quad (1.40)$$

This increases more rapidly ($\propto k_f$) for large k_f. Moreover, at normal density, with $a = 1{\cdot}61$, Ω_2' is between four and five times at large as Ω_2. The second-order theory has been extended to tensor forces by Levinger *et al.* (1960), who consider several possible shapes. The tensor force does not contribute in first order, vanishing when the spin average is performed. In the same way there are no cross terms in second order between central and tensor forces. However, the pure tensor contribution can in second order be very substantial. If a 'weak' tensor potential is used, i.e. one just sufficient to account for the observed stronger interaction in triplet than singlet states in the effective range region (assuming equal central forces), it gives an energy shift of about -5 MeV per particle. The low energy data, including the deuteron quadrupole moment, can equally be fitted by a 'strong' tensor force and a much weaker triplet central potential (with about 0·6 of the singlet strength); the tensor shift in second order is then about -10 MeV per particle. By contrast if no tensor force is allowed but the triplet central force increased appropriately (by a factor of about 1·4) the second-order shift per nucleon increases by only about -1 MeV. The first-order shift of course increases also, by about -9 MeV per particle. Thus tensor forces tend to dominate in the second-order term, and appreciably worsen the convergence.

Some discussion of third-order perturbation theory is contained in § 3.6. With central forces, according to de Dominicis and Martin (1957 a), the rate of convergence is maintained. The effect of tensor forces in third order terms has not been discussed.

1.5. *The Hard Core in the Many-body Problem*

Ordinary perturbation theory is inapplicable for realistic nuclear potentials because of the hard core; even if this were taken to be finite it would have to be so large that the convergence of the perturbation series would be very poor. Before introducing the two principal methods which have been used to handle this situation we note that, since the radius of the core ($\approx 0{\cdot}4f$) is small compared to the mean separation ($1{\cdot}6r_0 \approx 1{\cdot}8f$) of nucleons in the unperturbed fermi gas state, the effect of the core may, in some sense, be regarded as 'weak'. Thus we expect that, in spite of the strong potential in the core region, the system retains many of the features of the unperturbed state and, for example, remains more closely akin to a gas than to a liquid. This will be illustrated in § 2 where it is explicitly demonstrated that a hard core of very small radius has negligible influence on the properties of the system. There is some experimental support that this situation holds to a large extent in the nucleus from the fact that the mean free path of low energy nucleons incident on nuclei is known to be large compared to the mean separation. Thus the unperturbed fermi gas is a reasonable starting point.

The first method is based on the use of trial wave functions and the variation principle. As has been stated, the results of first-order perturbation theory can equally well be regarded as arising from the use of a determinantal trial wave function of plane wave states. With a hard core potential this only gives an upper limit of $(+\infty)$ on the value of the energy per particle. By relaxing the translational symmetry of the single particle states it is of course possible to obtain a more reasonable bound†—the individual nucleons would have to remain in spatially separated states so that their hard cores did not overlap. The considerations above would suggest that such a highly correlated wave function would be very unrealistic. An alternative is to multiply the original determinantal wave function by a factor which vanishes whenever any two particles approach within the hard core radius. The simplest form of this correlation factor is a product

$$\Pi f(r_{ij}) \qquad \qquad (1.41)$$

of factors for each interparticle distance, where f vanishes over the appropriate range. Outside this range f can be determined in principle by minimizing the energy. This is the method proposed by Jastrow (1955). It is not without difficulties in practice. It is the main subject of § 2.

The second method is a development of the perturbation rather than the variation approach. It can be introduced by noting that the second-order perturbation energy can be incorporated in the first order by replacing there the matrix elements of v by those of the operator

$$v + \sum_{\mathbf{l},\mathbf{l}'} \frac{v\,|\mathbf{l}\mathbf{l}'\rangle\langle\mathbf{l}\mathbf{l}'|\,v}{\mathscr{E} - l^2/2M - l'^2/2M} \qquad \qquad (1.42)$$

with $\mathscr{E} = (m^2 + m'^2)/2M$. This can be written more compactly

$$v + v\frac{Q}{\mathscr{E} - T}v \qquad \qquad (1.43)$$

where T is the kinetic energy operator and the two-body operator Q projects onto states above the fermi surface. Ignoring the exclusion principle as in obtaining (1·40) we would have instead

---

† That the Hartree–Fock state with least energy need not have translational symmetry has been stressed recently by Overhauser (1960). Our example shows that the lowest energy Hartree–Fock state is not necessarily then the best starting point for the theory. See also Kohn and Nettel (1960) and Broutt (1960), who suggest that Overhauser's specific proposals may not be relevant in three dimensions, and Henley and Willets (1961), Sawada and Fukuda (1961) and Bloch (1960 a).

$$v+v\frac{P}{\mathscr{E}-T}v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.44)$$

where P indicates the principal part. Now these are just the first terms in the expansion of the 'reaction matrix' of collision theory, closely related to the scattering operator (for a convenient introduction to these ideas see Brueckner 1958). The complete series is generated by the integral equation

$$K^0(\mathscr{E})=v+v\frac{P}{\mathscr{E}-T}K^0(\mathscr{E}). \quad . \quad . \quad . \quad . \quad . \quad (1.45)$$

The operator K^0 remains quite finite and well defined even when the potential has a hard core (the matrix elements are linear combinations of the tangents of the scattering phase shifts) although each term in its perturbation series becomes nonsense. Now when higher order terms in perturbation theory for nuclear matter are examined a sub-set is found which is generated by the equation

$$K(\mathscr{E})=v+v\frac{Q}{\mathscr{E}-T}K(\mathscr{E}). \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.46)$$

This is very like (1.45), and it can be hoped that there also the solution remains sensible for singular potentials. It represents roughly speaking the effect of treating two-body collisions in all orders of perturbation theory while ignoring many-body effects. The operator K, rather than v, is then a reasonable measure of the effective strength of the interaction, and it is reasonable to rearrange the whole series in terms of it. Of course such rearrangements of series with unexamined convergence properties are not mathematically respectable. We take the usual physicist's view that they are permissible until things go manifestly wrong—which they do. This approach, due mainly to Brueckner, is developed in § 3. What goes wrong is discussed in § 4.

§ 2. TRIAL WAVEFUNCTION METHODS

2.1. *Two-body Correlations and the Cluster Expansion*

Here we take up in detail the method of correlation factors mentioned briefly at the end of the last section. We introduce the trial wave function

$$\begin{aligned}\Psi&=\prod_{i>j} f(r_{ij})\,\Phi\\ &=F\Phi,\end{aligned} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.1)$$

where $r_{ij}=|\mathbf{r}_i-\mathbf{r}_j|$, Φ is the fermi gas state, a determinant of plane waves as discussed in § 1.2, and f is a function which is real (using time reversal invariance) and which satisfies

$$\left.\begin{aligned} f(r)&=0, \quad r<c,\\ f(r)&=1, \quad r>R. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.2)$$

The last condition is imposed to justify the neglect of surface integrals in partial integrations used to obtain the Euler equation. Subsequently we allow R to be arbitrarily large. The form for f given in (2.2) is appropriate for a nuclear potential containing a hard core with radius c in all states, as is usually assumed. We shall frequently denote $f(r_{ij})$ by $f(ij)$.

We require to determine $f(r)$ by minimizing $\bar{H}$, given by

$$\bar{H} = \int \frac{(F\Phi)^* H F\Phi}{(F\Phi)^* F\Phi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2.3)$$

with respect to variations in $f(r)$. In (2.3), and in future equations, we use the notation that, when the integration variables are not indicated explicitly, an integral sign stands for integration over all nuclear coordinates and summation over all nuclear spins. H is the complete hamiltonian for the nucleus. Unfortunately it is not possible to evaluate $\bar{H}$ in closed form and it is necessary to use some approximate method. This means that the variational principle, $E \leqslant \bar{H}$, is only approximately valid. It is important, therefore, that f should be varied within some restricted class of functions for which the approximate method of evaluating $\bar{H}$ is adequate, and, as we shall see, the omission of such a restriction can cause trouble. It is by no means certain that the restricted class is sufficiently large to allow a good estimate of E.

The approximation used to evaluate $\bar{H}$ is based on the expectation that the two-particle correlation, caused mainly by the hard core, disappears rapidly outside the range of the core, i.e. $|f^2(r)-1|$ tends rapidly to zero for $r > r_c$. This enables one to make an expansion in the range of the correlation function divided by the mean separation. Such an expansion, the 'cluster expansion', is familiar in the classical theory of imperfect gases. The method was first used in a quantum mechanical variational problem by Dingle (1949), who applied it to a hard sphere boson gas. Usually only the lowest order term is kept, which means that simultaneous correlation between three or more particles is ignored. Some indication that this might be a useful approximation is given by the fact that $(c/\bar{r}) \approx 0{\cdot}2$, so that, if f is taken to be zero inside the core and unity outside it, the higher terms of the cluster expansion should be small. Actually such an f is very unrealistic and we shall see that more reasonable forms give significant corrections.

The equation for $\bar{H}$ (2.3) can be written

$$\bar{H} = \frac{-\frac{1}{2}M \sum_i \int \Phi^* F\, \nabla_i^2 (F\Phi)}{\int |\Phi|^2 F^2} + \frac{\frac{1}{2}\sum_{i,j} \int \Phi^* F\, v(ij)\, F\Phi}{\int |\Phi|^2 F^2} \quad . \quad . \qquad (2.4)$$

where $v(ij)$ is the interaction between nucleons i and j. To consider the potential energy term it is convenient to write $v(ij)$ in the form

$$v(ij) = \sum_{n=1}^{4} v_n(r_{ij})\, O_n(ij) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2.5)$$

where $O_1 = 1$, $O_2 = \boldsymbol{\sigma}_i . \boldsymbol{\sigma}_j$, $O_3 = \boldsymbol{\tau}_i . \boldsymbol{\tau}_j$ and $O_4 = \boldsymbol{\sigma}_i . \boldsymbol{\sigma}_j \boldsymbol{\tau}_i . \boldsymbol{\tau}_j$. We do not include tensor forces since, with the wavefunction (2.1), they will not contribute to the energy. Then we can write for the potential term in (2.4)

$$\tfrac{1}{2}\rho A \int \sum_n v_n(r)\, G_n(r) f^2(r)\, d\mathbf{r},$$

where

$$G_n(\mathbf{r}) = \frac{A^2}{\rho^2} \int \frac{F^2}{f^2(12)} |\Phi|^2 O_n(12)\, \delta(\mathbf{r}_1 - \mathbf{r})\, \delta(\mathbf{r}_2) \Big/ \int |\Phi|^2 F^2 \tag{2.6}$$

and

$$\rho = (A/\Omega).$$

We use the equations

$$\int \Phi^* F\, \nabla_i^2 (F\Phi) = \tfrac{1}{2} \int [F^2\{\Phi^*\ \nabla_i^2 \Phi + \Phi\ \nabla_i^2 \Phi^*\} + 2|\Phi|^2 F \nabla_i^2 F + \nabla_i |\Phi|^2\, \nabla_i F^2],$$

$$\nabla_i F^2 = \sum_i \frac{\nabla_i f(ij)^2}{f(ij)^2} F^2,$$

and

$$\nabla_i^2 F = \sum_i \frac{\nabla_i^2 f(ij)}{f(ij)} F + \sum_j \nabla_i f(ij\, .\, \nabla_i \left(\frac{F}{f(ij)}\right),$$

to write the kinetic energy term in (2.4) as

$$\Big| A T_0 - \frac{\rho A}{2M} \int d\mathbf{r}\, \{(f\, \nabla^2 f) G_1 + \tfrac{1}{2} f\, \nabla f . \nabla G_1 + \tfrac{1}{2} f\, \nabla f . \mathbf{H}$$

where $T_0 (= 3k_f^2/10M)$ is the unperturbed kinetic energy per particle and

$$\mathbf{H}(\mathbf{r}) = \frac{A^2}{\rho^2} \int \frac{F^2}{f^2(12)} \delta(\mathbf{r}_1 - \mathbf{r})\, \delta(\mathbf{r}_2) \frac{\partial}{\partial \mathbf{r}_1} |\Phi|^2 \Big/ \int F^2 |\Phi|^2. \tag{2.7}$$

Integrating by parts and adding the potential energy we have finally

$$\bar{H}/A = T_0 + \frac{\rho}{2M} \int d\mathbf{r} \{G_1 (\nabla f)^2 + \tfrac{1}{2}(\nabla G_1 - \mathbf{H}) . \nabla f^2\} + \frac{\rho}{2} \sum_n \int d\mathbf{r}\, f^2 v_n G_n. \tag{2.8}$$

Provided the range of f is sufficiently small it will be a good approximation to replace F^2 and F^2/f^2 by unity in the equations for G and $\mathbf{H}$.

Then,
and

$$\left.\begin{aligned} \mathbf{H} &= \nabla G_1 \\ G_n &= d_n (k_f r) \end{aligned}\right\} \tag{2.9}$$

where

$$\left.\begin{aligned} d_1(x) &= 1 - \tfrac{1}{4} l^2(x), \\ d_2(x) &= d_3(x) = \tfrac{1}{3} d_4(x) = -\tfrac{3}{4} l^2(x), \end{aligned}\right\} \tag{2.10}$$

with

$$l(x) = 3x^{-3} (\sin x - x \cos x) \tag{2.11}$$

The first approximation to (2.8) is then

$$\bar{H}/A = T_0 + \rho \int d\mathbf{r} \left\{ \frac{d_1}{2M} (\nabla f)^2 + \tfrac{1}{2} \sum_n v_n d_n f^2 \right\}, \tag{2.12}$$

This is the first term in the general cluster expansion of Iwamoto and Yamada (1957 a). A different form is given by Hartogh and Tolhoek (1958) and by Aviles (1958), who develop a systematic expansion in powers of the density parameter k_f. To obtain the first term in this the functions d in (2.10) are replaced by their values at the origin $k_f r = 0$. The latter expansion converges very poorly for nuclear matter (Temkin 1960) because the force range is comparable with the internucleon spacing k_f^{-1}.

As a trivial application consider the pure hard core potential. Take a correlation function $f(r/c)$ which is zero for argument less than unity. The potential energy term vanishes exactly, and (2.12) gives

$$(\bar{H}/A) = T_0 + \frac{c\rho d_1(0)}{2M} \int dx\, x^2 \{f'(x)\}^2.$$

If the cluster expansion converges this is an upper limit for small c. For this purely repulsive potential T_0 is a lower limit. So we confirm formally that a sufficiently small hard core has negligible effect on the energy.

In the general case Edwards (1960, private communication) has found that the cluster development can be avoided and $(\bar{H}/A)$ evaluated exactly if the correlation factor Πf^2 is somewhat modified. This has the advantage of preserving the variational significance of the method. Unfortunately not enough correlation is retained to give finite results with hard core potentials.

2.2. *The Euler Equations for the Correlation Function*

Consider now the minimization of $\bar{H}/A$, given by (2.12), for arbitrary variation of f. Taking for simplicity a spin independent force, we must minimize

$$T_0 + \rho \int d\mathbf{r}\, d_1 \left\{ \frac{\nabla f)^2}{2M} + \tfrac{1}{2} v f^2 \right\}. \qquad (2.13)$$

The Euler equation is

$$-\frac{1}{Md_1} \nabla (d_1 \nabla f) + vf = 0, \qquad (2.14)$$

which is to be solved with $f(R) = 1$. When the restricting radius R is made arbitrarily large the solution has asymptotic form

$$f(r) - 1 \sim -\alpha/r \qquad (2.15)$$

where α is the 'scattering length' for the modified Schrodinger eqn. (2.14). Thus f has a long-range part which vitiates the cluster expansion. Nevertheless, (2.12) can still be justified as a consistent first approximation if α is small. For then a weak non-singular potential $v(r)$ can be chosen in many ways such that

$$v = v' + \delta v$$

where v' has zero scattering length. It is not hard to see that (2.12) is correct to lowest order in the range of v' and in the strength of δv.

The energy shift per particle can be given in terms of α. By partial integration in (2.13), using (2.14),

$$\Delta E/A = \rho\int d\mathbf{r}\,\nabla\left\{\frac{d_1}{2M}f\,\nabla f\right\}$$

$$= \rho\frac{4\pi}{2M}\,\underset{r\to\infty}{\mathrm{Lt}}\,(d_1\,f r^2 f')$$

$$= \rho\,(2\pi\alpha/M). \quad \text{. . . . .} \quad (2.16)$$

For the pure hard core (2.14) gives

$$f = \int_c^r \frac{ds}{s^2 d_1(k_f s)}\bigg/\int_c^\infty \frac{ds}{s^2 d_1(k_f s)}. \quad \text{. . . . . .} \quad (2.17)$$

For large r, to first order in c,

$$f = 1 - d_1(0)\,(c/r)$$

whence

$$E = \frac{2\pi\rho}{M}\frac{3c}{4}, \quad \text{. . . . .} \quad (2.18)$$

plus terms of higher order in c. For small c this is the correct result (Lenz 1929), although without further reasoning we could have been confident only of an upper limit. To obtain higher order terms cluster corrections must be added to (2.12). This has been considered by Temkin (1960) with the assumed form

$$f = 1 - \frac{c}{r}\exp\left[-\mu(r-c)\right] \quad \text{. . . . . . .} \quad (2.19)$$

for $r > c$. The best μ is density dependent, approaching zero (as k_f) at low density so that (2.19) then agrees with (2.15) near the core. Only a rather poor upper bound is obtained for the coefficient of the c^2 term (see § 3.5) and the dependence on the number of spin states is wrong. Temkin conjectures that the fault lies mainly not with the assumed form (2.19) but rather with the Jastrow wavefunction (2.1) itself, and that generalization of the latter along the lines of § 2.5 is necessary. This is in contrast with the boson case, where Iwamoto (1958) and Aviles (1958) obtained a close upper bound on the first correction ($\propto c^{5/2}$) to the Lenz term. In the boson case it is for this purpose necessary to consider not merely the next cluster term but an infinite subset of higher terms.

Returning to the general case, the following difficulty has been observed by Emery (1958). If the potential contains a sufficiently strong attraction outside the core, the solution of (2.14) does not minimize (2.13), which is, in fact, unbounded below. To see this we suppose the modified Schrödinger equation

$$-\frac{1}{M}\nabla(d_1\nabla\psi) + d_1 v\psi = \mathscr{E}\,d_1\psi \quad \text{. . . . .} \quad (2.20)$$

has a bound state ψ, with energy $\mathscr{E}<0$, which we normalize so that

$$\int |\psi|^2 d\mathbf{r} = 1.$$

Then, if f is any function satisfying the required boundary conditions, eqn. (2.2), $(f+B\psi)$ will also satisfy these conditions where B is any constant. Putting this correlation function in (2.13) and using (2.20) we find for the energy shift per particle

$$\Delta E(f+B\psi) = \Delta E(f) + (\text{constant}) \,.\, B + B^2\mathscr{E}.$$

Since $\mathscr{E}<0$ it follows by letting $B \to \infty$ that the energy can be made arbitrarily low. Of course no physical significance can be attached to this result since, with large B, the neglected cluster corrections will be large. The result shows, however, that, with a potential for which (2.20) has a bound state, the variational method has no value. According to Emery this difficulty occurs for any potential for which it is possible to obtain a negative energy shift in (2.13), but this is not in fact true. We can rearrange (2.20) into an ordinary Schrödinger equation with a effective potential

$$\left(v + \frac{1}{M}\frac{\nabla^2 d_1^{1/2}}{d_1^{1/2}}\right),$$

and the difficulty only arises when this effective potential has a bound state. The term depending on d_1 in this effective potential is fairly small, but as it is not of constant sign it is not possible to relate this condition exactly to the condition for v to have a bound state. With a realistic nuclear potential it is unlikely that v does have a bound state, since the existence of the deuteron probably depends essentially on the tensor force, which does not contribute in (2.12).

However, even if the Emery difficulty does not occur, it seems likely that the long-range part of the correlation function, determined from (2.14) will be too large for the first approximation (2.12) to be adequate. This is certainly so if we ignore the d function since in this case α in (2.15) is the free space scattering length which, in the singlet state, is about $-24f$. This may be somewhat reduced by the proper inclusion of the d functions but we know of no detailed investigation (see the following section, however).

2.3. *Numerical Calculations*

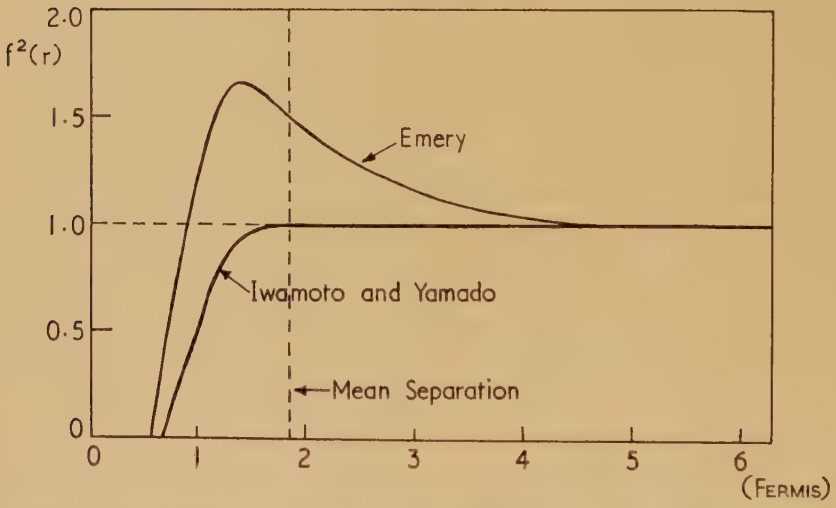
In this section we review the numerical calculations which have been made using the ideas of the previous sections of this chapter. The only realistic calculations which we know of using the Euler equation have been made by Emery (unpublished). Emery used the central part of the Brueckner–Gammel potential (§ 1.3) acting in even states only, and solved the Euler equation, i.e. (2.14) generalized to allow for a spin dependent force.

The results at three densities are given in table 1. The binding energy at equilibrium density is approximately 23 MeV per particle, corresponding to $k_f \approx 1{\cdot}6 f^{-1}$. According to the work of Brueckner and Gammel (§ 3.8) the potential used by Emery should not give any binding energy (Brueckner and Gammel obtain about 20 MeV per particle from the tensor force), so it seems that the first approximation is very bad. In view of the surprisingly small scattering length this conclusion is unexpected, and some further work on this point would be useful.

Table 1. Results obtained by Emery for the solution of the Euler equation

k_f	Binding energy per particle	Scattering length
$1{\cdot}3 f^{-1}$	20·9 MeV	$-1{\cdot}08 f$
$1{\cdot}5 f^{-1}$	22·8 MeV	$-0{\cdot}85 f$
$1{\cdot}7 f^{-1}$	22·5 MeV	$-0{\cdot}68 f$

Fig. 1



Showing the correlation functions in the variational calculations of Iwamoto and Yamado (1957) and Emery (1958).

Other calculations have been done with restricted forms of f, in which only a few parameters are determined by the variation process. If the cluster corrections were negligible this would give an upper bound for the energy, although of course it might not be a close bound. The main interest of such calculations is in the magnitude of the corrections.

Iwamoto and Yamada (1957 b) used the form

$$\begin{aligned} f(r) &= 0, \quad r \leqslant c \\ &= 1 - \exp[-\beta(r-c)], \quad r > c. \end{aligned}$$

Their potential, fitted to low energy singlet scattering, is a hard core of radius $c = 0{\cdot}6f$ plus an exponential Serber attraction. The only variational parameter is β, for which the optimum value is $5f^{-1}$ at an equilibrium density with $r_0 = 1{\cdot}1f$, when the energy per particle is $-5{\cdot}2$ MeV. That the energy is too high is to be expected in a method which suppresses the tensor force (see § 3.8). With the quoted value of β the correlation seems of quite short range compared with the particle spacing. Nevertheless, the first cluster corrections, estimated by Iwamoto and Yamada, are substantial. The energy shift per particle in lowest order consists of 48 MeV kinetic energy and 77 MeV potential energy. The first corrections to these are respectively (-12) MeV and $(+12)$ MeV, or about 20%. The numerical approximations used do not permit a significant estimate of the percentage correction to the total energy.

In this connection the work of Dabrowski (1958 a, b, c) is of interest, although concerned with the particular nucleus ^{16}O rather than with nuclear matter. He used potentials, fitted to low energy singlet scattering, which had Yukawa–Serber attractions in addition to hard cores. For f he chose a gaussian form outside the core. The results are shown in table 2 for the two values of core radius considered. They should be compared with the experimental values $r_0 = 1{\cdot}1f$, $E = -128$ MeV. These should in fact be reasonably approximated for some intermediate core radius. The first cluster corrections (evaluated for $c = 0{\cdot}6f$) to the separate kinetic and potential energies are relatively smaller than those of Iwamoto and Yamada, but the percentage correction to the total energy is still large, being about 40%.

Table 2. Showing the results of Dabrowski's (1958 a, b, c) calculations on ^{16}O. The fourth, fifth and sixth columns give the results of the lowest order calculation, and the last three columns give the next order corrections. Energies are in MeV and lengths in fermis

c	β	r_0	Kinetic energy	Potential energy	Total energy	Kinetic energy correction	Potential energy correction	Total correction
0·2	0·75	0·86	984	−1183	−199	—	—	—
0·6	2	1·5	651	−716	−65	6	−31	−25

In the calculations mentioned the correlation functions have the property $f \leqslant 1$ everywhere. Thus any degree of positive correlation, such as might naturally be expected from the attractive part of the potential, is excluded. Emery (1958) used a form free from this restriction, namely

$$f(r) = 0, \qquad r \leqslant c,$$

$$f(r) = \left\{1 - \frac{c}{r}\exp\left[-\lambda(r-c)\right]\right\}\left[1 + \alpha\exp(-\beta r)\right], \quad r > c$$

where α, β and λ are variational parameters. His potential was different in singlet and triplet states and fitted separately to both sets of low energy data, despite the omission of a tensor force. He chose $c = 0{\cdot}5f$. In lowest order the resulting energy and density were close to the required values. However, because of the strong positive correlation, the first cluster correction was large and was estimated to contribute about 20 MeV per particle to the binding energy. The rapid convergence of the perturbation expansion with non-singular potentials (§ 1.4) suggests that the large positive correlation found by Emery is probably spurious (in fact with the potential used it may be, as discussed in § 2.2, that the first approximation has no true minimum), but the calculation shows the difficulty encountered in obtaining a reasonable value for the energy by this method. Unfortunately there are no results available to show whether the inclusion of the next order term appreciably alters the position of the minimum.

The overall situation is thus rather unsatisfactory. The work that has been done with assumed forms for f suggests that cluster corrections are essential for accurate results, and may even be so large as to vitiate the whole method. Certainly the labour involved increases enormously when it is required to calculate even the next order term in the cluster expansion. Finally it should be noted that for realistic results it is certainly necessary to generalize the trial wavefunction (2.1), at least to include tensor correlations which are required in order to obtain the contribution of the tensor force to the binding energy.

2.4. *The Effective Potential*

We have seen that, whilst it might be possible to treat at least the most important part of the correlation due to the hard core by the cluster expansion method, any positive correlation due to the attractive part of the potential is more troublesome. It is suggested therefore (e.g. Clark and Feenberg 1959) that one might use an F factor to remove the hard core (2.1) and then treat the remaining part of the potential by some other method, e.g. by perturbation theory. To see how this might be done we consider a complete set of unperturbed (fermi gas) states of the nucleus Φ_α, and define the set Ψ_α by

$$\Psi_\alpha = F\Phi_\alpha. \qquad \ldots \ldots \ldots \quad (2.21)$$

Then, in lowest order of the cluster expansion†, we obtain

$$\frac{[\langle\Psi_\alpha|\Psi_\alpha\rangle]^{1/2}[\langle\Psi_\beta|\Psi_\beta\rangle]^{1/2}}{\langle\Psi_\alpha|H|\Psi_\beta\rangle}=T_\alpha\langle\Phi_\alpha|\Phi_\beta\rangle + \sum_i\sum_j\int\Phi_\alpha{}^*[1/2M\{\nabla f(ij)\}^2+\tfrac{1}{2}v(ij)f^2(ij)]\Phi_\beta \quad . \quad . \quad . \quad (2.22)$$

where T_α is the unperturbed kinetic energy of the state Φ_α. Thus, in this order, we can ignore the F factor provided we replace the potential $v(ij)$ by an *effective potential* given by

$$v_{\mathrm{eff}}=\frac{1}{M}(\nabla f)^2+vf^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.23)$$

The states Ψ_α defined by (2.21) are in general non-orthogonal but this does not show up in the lowest order approximation. The effective potential will be non-singular and can be used in low order perturbation calculations (cf. § 1.4) and, in finite nuclei, in shell model calculations for example.

The function f must be chosen to satisfy, as well as is possible, two conditions; (i) it must be such as to make the higher order cluster terms negligible; and (ii), it must be such that the approximations used to treat v_{eff} are adequate, i.e. if perturbation theory is used then v_{eff} must be sufficiently 'weak'. In practice it is far from clear, however, how one should apply these conditions. One possibility, which is of interest because it gives results which are related to those obtained by direct perturbation theory methods (see § 3), is obtained in the following way (Clark and Feenberg 1959 and Moszkwoski and Scott 1960). Consider a potential v interacting only in s-states and separate it into two parts, a long-range and a short-range part, defined by

$$\left.\begin{array}{lll} v_{\mathrm{l}}=0 \text{ and } v_{\mathrm{s}}=v, & r\leqslant d, \\ v_{\mathrm{l}}=v \text{ and } v_{\mathrm{s}}=0, & r>d. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.24)$$

The distance d is chosen so that the solution of

$$-\frac{1}{M}\nabla(d_1\nabla f)+\sum_n v_{\mathrm{s}n}d_{\mathrm{n}}f=0, \quad . \quad . \quad . \quad . \quad (2.25)$$

where the v_{sn} are the short-range parts of the v_n (2.5), defined in an analogous way to v_{s} in (2.24), are equal to 1 for $r>d$. With this f the effective potential is given by

$$\left.\begin{array}{ll} v_{\mathrm{eff}}=v_{\mathrm{l}}, & r>d \\ \phantom{v_{\mathrm{eff}}}=\dfrac{1}{M}(\nabla f)^2+v_{\mathrm{s}}f^2, & r\leqslant d. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.26)$$

---

† Note that since we are not merely considering expectation values in the ground state we are here using the term 'cluster expansion' in a generalized sense. For reasonably low excited states the conditions for the first approximation to be adequate should be approximately as before.

Equation (2.25) is the Euler equation for the minimization of $\bar{H}/A$ (2.12) when the potential is just v_s. The energy shift to lowest order in v_{eff} is immediately seen to be just the expectation value of v_l in the state Φ. In second order there are terms in v_l^2 and also contributions from the part of v_{eff} with $r \leqslant d$.

If we now make the approximation of putting $l(x) = 0$ in (2.25), the equation becomes the zero energy Schrödinger equation for the potential v_s. Thus v_s has been chosen so that, approximately, it produces no s wave scattering at zero energy. The value of d is approximately $1f$, so the neglect of the higher order cluster terms is not unreasonable.

Moszkowski and Scott (1960) use a method which is equivalent to a generalization of the above procedure, in which the f's depend on the state of the particles concerned (see § 2.5). The distance d is then a function of momentum defined so that v_s, in the modified Schrödinger equation, gives no s-phase shift at any energy. Second-order perturbation theory is used; the method then becomes equivalent to the Brueckner method (§ 3) with a certain approximation to the solution of the K matrix equation, and we shall therefore treat it further in § 3.5. We note however that, from the viewpoint of this chapter, the choice of d, and hence of f, is rather arbitrary. Ideally one should choose d so that the sum of the neglected cluster terms and the corrections to the method used to treat v_l is a minimum, but this is hardly practical. Allowing d to become a function of momentum, as above, certainly has the effect of weakening v_l as the momentum increases (d is an increasing function of momentum) and hence improving the convergence of the perturbation expansion in v_l, as pointed out by Moszkowski (1959), but it is not obvious that this is not at the expense of increasing the magnitude of the neglected cluster terms.

2.5. *The Bethe–Goldstone Equation*

The physical ideas underlying the methods of this section and those used to justify the methods of § 3, are not dissimilar, and there has been some interest in the relationship of the two methods. It has not been found possible to obtain the approximation of § 3 by a variational method, but we shall indicate here how this approximation arises from a generalization of (2.1) by the neglect of certain cluster terms. Some of the ideas of this section have been used by Leigh (1958), by Weisskopf and de Shalit (1959), and by Dabrowski (1958 c). How far one can regard this section as a 'derivation' or merely as a series of steps leading to a known result we leave to the readers.

It is necessary to generalize the wavefunction (2.1) by allowing the internucleon correlations to depend on the states occupied by the nucleons. We denote the single particle (plane wave) states by $\phi_i\,(\mathbf{r}_i)$ and define the non-antisymmetrized ground state ϕ by

$$\phi = \prod_{\substack{\text{occupied}\\ \text{states}}} \phi_i(\mathbf{r}_i). \qquad \qquad (2.27)$$

Then we define an antisymmetrizing operator $\mathscr{A}$ by

$$\Phi = \mathscr{A}\phi, \quad \mathscr{A}^2 = \mathscr{A}. \qquad (2.28)$$

The generalization of (2.1) is obtained by putting

$$\Psi = \mathscr{A} F \phi, \qquad (2.29)$$

with

$$F = \prod_{i>j} f_{ij}(r_{ij}), \qquad (2.30)$$

$$f_{ij} = f_{ji}. \qquad (2.31)$$

We thus have $\frac{1}{2}N^2$ different functions f_{ij}. We require these to be such that

$$H\Psi = E\Psi + \text{higher order cluster terms.} \qquad (2.32)$$

To see how this can be done we have

$$\begin{aligned} H\Psi &= \mathscr{A} H F \phi \\ &= \mathscr{A}\left\{T_0 A - \tfrac{1}{2}\sum_{ij}\frac{1}{f_{ij}}\left[\frac{1}{M}\nabla^2 f_{ij} + \frac{i}{M}(\mathbf{k}_i - \mathbf{k}_j) . \nabla f_{ij} - v(ij) f_{ij}\right]\right. \\ &\qquad \left. - \frac{1}{2M}\sum_{ijk}\frac{\nabla f_{ij} . \nabla f_{ik}}{f_{ij} f_{ik}}\right\} F\phi. \qquad (2.33) \end{aligned}$$

We are here denoting the momentum of the state ϕ_i by $\mathbf{k}_i$. Each gradient operator acts on the argument of the f factor immediately following it. The f_{ij} are then chosen to satisfy the two-body equations

$$\left\{\Delta e(ij) f_{ij} + \frac{1}{M}\nabla^2 f_{ij} + \frac{i}{M}(\mathbf{k}_i - \mathbf{k}_j) . \nabla_i f_{ij} - Q_{ij}' v(ij) f_{ij}\right\} \mathscr{A}_{ij}\phi_i(i)\phi_j(j) = 0 \qquad (2.34)$$

where $\mathscr{A}_{ij}$ is the operator which antisymmetrizes between particles i and j, and where Q_{ij}' is a projection operator which we discuss further below. The necessity for the f_{ij} to depend on the states of the two particles is seen immediately from this equation, as pointed out by Dabrowski (1958 c). Equation (2.33) becomes

$$H\Psi = \left[AT_0 + \tfrac{1}{2}\sum_{ij}\Delta e(ij)\right]\Psi - \mathscr{A}\left[\tfrac{1}{2}\sum_{ij}(Q'_{ij} - 1)v(ij) + \frac{1}{2M}\sum_{ijk}\frac{\nabla f_{ij} . \nabla f_{ik}}{f_{ij} f_{ik}}\right] F\phi. \qquad (2.35)$$

If we can neglect the second bracket then we obtain, as our approximation to the energy

$$E/A = T_0 + \frac{1}{2A}\sum_{ij}\Delta e(ij). \qquad (2.36)$$

Returning now to the two-body eqn. (2.34) we introduce the two-body wavefunction

$$\psi_{ij}(ij) = \mathscr{A}_{ij} f_{ij}(r_{ij})\,\phi_i(\mathbf{r}_i)\,\phi_j(\mathbf{r}_j), \qquad (2.37)$$

then (2.34) becomes

$$\Delta e(ij)\psi_{ij} = -\left(\frac{1}{M}\nabla^2 + e(ij) - Q_{ij}'\,v(ij)\right)\psi_{ij} \qquad (2.38)$$

where ∇ acts on the relative coordinate $\mathbf{r}_{ij}$ and where

$$e(ij) = \frac{1}{4M} |\mathbf{k}_i - \mathbf{k}_j|^2,$$

the relative kinetic energy of the interacting nucleons. We now make explicit use of the well-known suppression of free scattering in the interior of a fermi gas by the exclusion principle (Weisskopf 1950) by choosing Q_{ij}' so that $Q_{ij}'\, v(ij)$ does not scatter into occupied states. Thus we put

$$\begin{aligned} \langle \phi_\alpha(i)\, \phi_\beta(j)\, | Q_{ij}' &= 1 \text{ if } k_\alpha^2 \text{ and } k_\beta^2 > k_f^2 \\ &= 1 \text{ if either } \alpha = i \text{ and } \beta = j \text{ or } \alpha = j \text{ and } \beta = i, \\ &= 0 \text{ otherwise.} \qquad \qquad (2.39) \end{aligned}$$

It follows that $(Q_{ij}' - 1)v$ only scatters into states which are already occupied by particles other than i and j. Hence

$$(Q_{ij}' - 1)\, v(ij) f_{ij}\, \phi = 0, \qquad (2.40)$$

so that the term containing $(Q_{ij}' - 1)$ in (2.35) is of second and higher order in the correlation. This is also true of the remaining term in (2.35), so the use of (2.36) for the lowest order expression for the energy shift is plausible. It should be noted that these 'cluster corrections' are here wavefunctions, rather than expectation values in the unperturbed ground state, so the justification of their neglect is less clear here than previously.

From (2.38) the energy shifts are given by

$$\Delta e(ij) = \langle \phi_{ij}\, | v(ij)\, | \psi_{ij} \rangle \qquad (2.41)$$

where

$$\phi_{ij} = \mathscr{A}_{ij}\, \phi_i \phi_j, \quad \langle \phi_{ij} | \phi_{ij} \rangle = 1, \qquad (2.42)$$

and

$$\psi_{ij} = \phi_{ij} + \frac{Qv(ij)}{e(ij) + \Delta e(ij) - (1/M)\nabla^2} \psi_{ij} \qquad (2.43)$$

where Q is the projection operator onto states above the fermi surface. It follows from (2.43) that $\Delta e(ij)$ is $O(1/A)$ for a large system. The total energy shift, from (2.38), is then $O(A)$ as required. In the infinite limit $\Delta e(ij)$ disappears from (2.43) and the equation is then, apart from one aspect mentioned in the next paragraph, identical to the *Bethe–Goldstone equation* (Bethe and Goldstone 1957) which can be obtained by a partial summation of perturbation theory diagrams (see § 3.4). The energy shifts $\Delta e(ij)$ are (apart from the point discussed in the next paragraph) given by the same perturbation expansion as the diagonal matrix elements of the Brueckner K matrix. We shall not therefore discuss the solution of this equation any further here. We note however that the Q' operator was chosen to some extent arbitrarily and that the choice made is a good one only if the solutions of (2.43) are reasonable. In fact it turns out that the solution to (2.43) has an unusual behaviour in the neighbourhood of the fermi surface (i.e. $k_i \approx k_j \approx k_f$) for potentials of interest and that the neglect of $\Delta e(ii)$ in (2.43) leads to a singular expression for the total energy shift.

This suggests that the choice of Q_{ij}' may not be suitable. The problem of singularities is discussed further in § 4.

There is an important difference between the theory of this section and that usually used in perturbation theory methods—we have here nothing analogous to the self-consistent energy denominators (§ 3.7), the $e(ij)$ in (2.43) being just kinetic energies. The fact that self-consistency effects seem to be important in nuclei (§§ 3.6, 3.7) implies a serious defect in the method of this section and indeed of this chapter (with the exception of § 2.4). In order to obtain self-consistency effects it is necessary to go beyond the lowest order in the cluster expansion and it seems to be necessary to introduce correlation functions f which are non-commuting operators (so that the correlation between two nucleons depends on their instantaneous states rather than on the states they occupy in the fermi gas). The task of dealing with such correlation functions appears to be very formidable.

Finally we mention the work of Tagami (1960) who used a wavefunction of the form (2.30), and calculated the expectation value of H in lowest order of the cluster expansion (i.e. the procedure of § 2.1 generalized to allow for the state dependence of the f_{ij}). He did not minimize this quantity but chose the f_{ij} to have general features suggested by the Bethe–Goldstone equation. With a potential containing a hard core and central Serber attraction, chosen to fit both singlet and triplet low energy data, he obtained a binding energy per particle of 18·3 MeV at an equilibrium value of r_0 equal to 0·94f. He evaluated the next order cluster corrections and found them to be small. With a similar potential, however, Moszkowski and Scott (1960), using methods described in § 3.5, found very different results—much too low energy and no equilibrium at a reasonable density.

§ 3. BRUECKNER THEORY

3.1. *Introduction*

The qualitative ideas of Brueckner theory have been described in § 1. They will now be developed in a formal way by rearrangement of a complete perturbation expansion. The ordinary time independent perturbation method is not suitable here (§ 3.2). The necessary time dependent formalism is outlined in § 3.3. Then the simplest form of Brueckner theory is stated and corrections to it discussed. In the light of these the 'self-consistent' theory is introduced, and the work of Brueckner and Gammel described. Some questions, relating to 'single particle energies' are discussed in § 3.9 and § 3.10, and finally some remarks made on the applicability of the rearrangement method.

Our account is based entirely on the 'linked-diagram' expansion. The simpler versions of the theory have also been obtained by rearrangement of ordinary perturbation theory (Rodberg 1957, Tobocman 1957) but this approach is dependent on the linked diagram demonstration that certain difficulties (§ 3.2) cancel. These same difficulties were inherent in the many early non-perturbative developments (Brueckner *et al.* 1954, Brueckner

1955 a, Brueckner and Levinson 1955, Eden and Francis 1955, Eden 1956, Watson 1956, Bethe 1956, Brenig 1957, and Kummel 1957). An important current approach to many-body problems is by way of Green's functions† (see for example Martin and Schwinger 1959 and Klein 1961). These occur incidentally in the present work (§3.10), but the infinite chain of coupled equations satisfied by them can be made the basis of the theory. The characteristc approximation then is to break off the infinite chain at some point, near the beginning. This is only roughly equivalent to natural partial summations in perturbation theory (see Martin and Kadanoff 1961). The method has not been applied in a practical way to the nuclear matter ground state‡, and is not developed here.

3.2. *Perturbation Theory for Many-body Systems*

The characteristic difficulty in the quantum perturbation theory of many-body systems can be explained as follows. Consider a large number N of similar non-interacting systems. Suppose that to the hamiltonian of each a perturbation proportional to a parameter λ is added, the systems remaining uncoupled. For each system the projection of the perturbed on the unperturbed wave function is of the form $(1-c\lambda^2)$ for small λ. For all N systems considered as a compound system the projection of the perturbed on the unperturbed state is then

$$(1-c\lambda^2)^N \approx \exp(-Nc\lambda^2) \approx 1-Nc\lambda^2+\tfrac{1}{2}N(N-1)c^2\lambda^4+\ldots.$$

Even for a small perturbation the successive terms increase rapidly for large N, and individual terms have no limit as $N\to\infty$. At the same time the perturbation series for independent component systems may converge very well. Now in dealing with a large volume of nuclear matter we may (very loosely) think of it as made up of very many smaller volumes which interact only weakly. Then it can be expected that perturbation expansions of some quantities will contain terms which have no limit for infinitely large volume. On the other hand, expansions for local quantities, such as energy density, should remain sensible. The problem is to extract these expansions from formalisms that start always with the state function of the whole system.

When the Rayleigh–Schrödinger or the Brillouin–Wagner perturbation method is used (Brueckner 1958 makes a detailed study) terms appear in each order involving various powers of the total particle number. It is only after extensive cancellation among these that sensible results emerge. If only the first few terms of the expansion are required this is no great inconvenience, but for an analysis of the whole perturbation series some method of generating the sensible terms directly is desirable. Such a method exists already in field theory, which is concerned always with systems of infinitely many degrees of freedom. As applied to the present subject the essential ideas were set out by Salam (1953). We follow below the later independent approach of Goldstone (1957).

---

† The method of Coester and Kümmel (1960) is somewhat related.

‡ See however the recent work of Puff (1961), *Annals of Physics*, 13, 317.

3.3. *The Linked Diagram Series*

Let H, E and $|\mathbf{0}\rangle$ be the hamiltonian, ground state energy and ground state of the system respectively, and let H_0 T_0 and $|0\rangle$ be the corresponding unperturbed quantities. The perturbation we denote by $V=H-H_0$. Then the energy shift due to the perturbation is

$$\Delta E=E-T_0=\langle 0|H-H_0|\mathbf{0}\rangle/\langle 0|\mathbf{0}\rangle$$
$$=\frac{\langle 0|V|\mathbf{0}\rangle}{\langle 0|\mathbf{0}\rangle}. \quad . \quad . \quad . \quad . \quad (3.1)$$

To obtain $|\mathbf{0}\rangle$ we use the time dependent S matrix defined by

$$S(t,t')=\exp(iH_0t)\exp[-iH(t-t')]\exp(-iH_0t'). \quad . \quad . \quad (3.2)$$

Then

$$\frac{|\mathbf{0}\rangle}{\langle 0|\mathbf{0}\rangle}=\underset{t\to-\infty(1-i\epsilon)}{\mathrm{Lt}}\frac{S(0,t)|0\rangle}{\langle 0|S(0,t)|0\rangle} \quad . \quad . \quad . \quad . \quad . \quad (3.3)$$

where ϵ is some positive number. To see this we have, in terms of the complete set of eigenstates $|\mathbf{n}\rangle$ of H with energies $E_{\mathbf{n}}$,

$$S(0,t)=\sum_{\mathbf{n}}\exp(iE_{\mathbf{n}}t)|\mathbf{n}\rangle\langle\mathbf{n}|\exp(-iH_0t),$$

whence the right-hand side of (3.3) is

$$\underset{t\to-\infty(1-i\epsilon)}{\mathrm{Lt}}\frac{\sum_{\mathbf{n}}|\mathbf{n}\rangle\exp[i(E_{\mathbf{n}}-T_0)t]\langle\mathbf{n}|0\rangle}{\sum_{\mathbf{n}}|\langle 0|\mathbf{n}\rangle|^2\exp[i(E_{\mathbf{n}}-T_0)t]}. \quad . \quad . \quad . \quad (3.4)$$

In the limit the terms in (3.4) with the lowest value of $E_{\mathbf{n}}$ (i.e. $E_{\mathbf{n}}=E_0=E$) dominate, so, provided A is such that the unperturbed ground state is non-degenerate (a 'closed-shell') and provided $\langle 0|\mathbf{0}\rangle\neq 0$, the result follows.

From (3.2) the S matrix satisfies

$$\frac{d}{dt}S(t,t')=-i\,V(t)\,S(t,t') \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.5)$$

where

$$V(t)=\exp(iH_0t)\,V\exp(-iH_0t). \quad . \quad . \quad . \quad . \quad . \quad (3.6)$$

Operators transformed as in (3.6) are said to be in the *interaction representation.* With the initial condition $S(t't')=1$, the solution of (3.5) by iteration gives

$$S(t,t')=1+(-i)\int_{t'}^{t}dt_1\,V(t_1)+(-i)^2\int_{t'}^{t}dt_1\int_{t'}^{t_1}dt_2\,V(t_1)\,V(t_2)+\dots$$
$$=\sum_n\frac{(-i)^n}{n!}\int_t^{t'}dt_1\,dt_2\dots dt_n\,T(V(t_1)\,V(t_2)\dots V(t_n)). \quad . \quad . \quad (3.7)$$

In (3.7) all time integrations are over the same interval, the symbol T denotes that the $V(t)$ must be rearranged in order from left to right of decreasing t and the $1/n!$ factor compensates for the fact that the same contribution is counted $n!$ times in the integral.

Then, from (3.1), (3.3) and (3.7)

$$\Delta E = \underset{t\to-\infty(1-i\epsilon)}{\mathrm{Lt}} \frac{\langle 0|\sum \frac{(-i)^n}{n!}\int_t^0 dt_1 \ldots dt_n\, T(V(0)\, V(t_1) \ldots V(t_n)\,|0\rangle}{\langle 0|\sum \frac{(-i)^n}{n!}\int_t^0 dt_1 \ldots dt_n\, T\,(V(t_1) \,\ldots\ldots\, V(t_n))\,|0\rangle}. \qquad \ldots \quad (3.8)$$

We now use the notation of second quantization. The unperturbed hamiltonian is written

$$H_0 = -\frac{1}{2M}\int d\mathbf{r}\, \psi^+(\mathbf{r})\, \nabla^2 \psi(\mathbf{r}), \qquad \ldots\ldots \quad (3.9)$$

and the perturbation

$$V = \tfrac{1}{2}\int d\mathbf{r}\, d\mathbf{r}'\, \psi^+(\mathbf{r})\, \psi^+(\mathbf{r}')\, v(\mathbf{r}-\mathbf{r}')\, \psi(\mathbf{r}')\, \psi(\mathbf{r}). \qquad \ldots \quad (3.10)$$

The latter form is for a local interaction, but the generalization is trivial. The *field operators* are defined by

$$\left.\begin{aligned} \psi(\mathbf{r}) &= \sum_{\mathbf{k}} u_{\mathbf{k}}(\mathbf{r})\, a_{\mathbf{k}}, \\ \psi^+(\mathbf{r}) &= \sum_{\mathbf{k}} u_{\mathbf{k}}{}^*(\mathbf{r})\, a^+{}_{\mathbf{k}}. \end{aligned}\right\} \qquad \ldots\ldots \quad (3.11)$$

The $u_{\mathbf{k}}$ are a complete set of one particle wave functions, with kinetic energy eigenvalues $k^2/2M$, with periodic boundary conditions in a cube of volume Ω. For spinless particles they are just plane waves

$$u_{\mathbf{k}} = \Omega^{-1/2} \exp(i\mathbf{k}\,.\,\mathbf{r}), \qquad \ldots\ldots \quad (3.12)$$

with each component of $\mathbf{k}$ an integral multiple of $2\pi\Omega^{-1/3}$. For particles with spin and isospin we consider the label $\mathbf{k}$ to specify a spin state as well as a momentum, and the $u_{\mathbf{k}}$ to contain also spin wave functions. The states $u_{\mathbf{k}}$ and operators ψ then have spinor indices which we normally suppress. In (3.10) for example v becomes a matrix in spin space and there are suppressed summations over the spinor indices of the field operators. The *creation* and *destruction* operators, $a_{\mathbf{k}}{}^+$ and $a_{\mathbf{k}}$, satisfy the anticommutation relations

$$\left.\begin{aligned} [a_{\mathbf{k}}, a_{\mathbf{k}'}]_+ &= [a_{\mathbf{k}}{}^+, a_{\mathbf{k}'}{}^+]_+ = 0, \\ [a_{\mathbf{k}}, a_{\mathbf{k}'}{}^+]_+ &= \delta_{\mathbf{k}\mathbf{k}'} \end{aligned}\right\} \qquad \ldots\ldots \quad (3.13)$$

A destruction operator acting on the vacuum gives zero. The many-body unperturbed ground state is

$$|0\rangle = \Pi\, a_{\mathbf{m}}{}^+\,|\mathrm{vacuum}\rangle, \qquad \ldots\ldots \quad (3.14)$$

where as before the label $\mathbf{m}$ indicates a state below the fermi surface.

Finally we introduce the interaction representation field operators

$$\left.\begin{aligned}\psi(\mathbf{r},t)&=\exp(iH_0t)\,\psi(\mathbf{r})\exp(-iH_0t)=\sum_{\mathbf{k}}\exp(-itk^2/2M)\,u_{\mathbf{k}}\,a_{\mathbf{k}},\\ \psi^+(\mathbf{r},t)&=\exp(iH_0t)\,\psi(\mathbf{r})\exp(-iH_0t)=\sum_{\mathbf{k}}\exp(itk^2/2M)\,u_{\mathbf{k}}{}^*\,a_{\mathbf{k}}{}^+.\end{aligned}\right\}\quad(3.15)$$

The final expressions here follow from (3.11) and (3.13) with (3.9) in the form

$$H_0=\sum_{\mathbf{k}}(k^2/2M)\,a_{\mathbf{k}}{}^+a_{\mathbf{k}}.$$

The interaction $V(t)$ can now be written

$$V(t)=\tfrac{1}{2}\int d\mathbf{r}\,d\mathbf{r}'\,\psi^+(\mathbf{r},t)\,\psi^+(\mathbf{r}',t')\,v(\mathbf{r}-\mathbf{r}')\,\psi(\mathbf{r},t)\,\psi(\mathbf{r},t).\qquad(3.16)$$

With (3.16) substituted in (3.9), evaluation proceeds by means of Wick's theorem (see, for example, Hamilton 1959). To state the result it is convenient to regard the time arguments of the ψ in (3.16) as infinitesimally earlier than those of the ψ^+. Then the symbol T in (3.8) can be regarded as specifying also the order in which field operators are to be taken within a single $V(t)$. It is also convenient to extend the definition of T to imply a change of sign when an odd permutation of fermion operators is required in passing from the written to the temporal order. Then the expectation value of a T-product of field operators is a sum of terms corresponding to all possible ways of associating in pairs the ψ operators with the ψ^+ operators. If there are n operators of each kind there are $(2n)!$ ways of pairing. The contribution of each such pairing is the product

$$\langle 0\,|\,T(\psi(\mathbf{r},t)\,\psi^+(\mathbf{r}',t'))\,|\,0\rangle\langle 0\,|\,T(\psi(\mathbf{r}'',t'')\,\psi^+(\mathbf{r}''',t'''))\,|\,0\rangle\ldots\qquad(3.17)$$

of the expectation values of individual pairs, multiplied by a factor (-1) if the written order of operators in (3.17) differs by an odd permutation from the original written order. The various contributions to the numerator and demoninator in (3.8) for a given value of n are conveniently represented by diagrams constructed as follows. We draw n vertical dotted lines labelled 1 to n to represent the n operators $V(t_1)\ldots V(t_n)$, and, for the numerator, a further dotted line labelled 0 to represent $V(0)$. We refer to these lines as 'interactions'. Then we join the ends of these interactions by directed solid lines such that one line enters and one line leaves each end of every direction. Each of these lines represents the pairing of two field operators. There are $(2n)!$ different ways of doing this for the denominator and $(2n+2)!$ ways for the numerator. The contribution of any diagram obtained in this way is

$$\frac{1}{n!}\int_t^0 dt_1\ldots dt_n\int d\mathbf{x}_1\,d\mathbf{y}_1\ldots d\mathbf{x}_n\,d\mathbf{y}_n\,I\,(\mathbf{x}_1,\mathbf{y}_1,t_1;\ldots\mathbf{x}_n,\mathbf{y}_n,t_n)\qquad(3.18)$$

for the denominator and

$$\frac{1}{n!}\int^0 dt_1\ldots dt_n\int d\mathbf{x}_0\,d\mathbf{y}_0\,d\mathbf{x}_1\ldots d\mathbf{y}_n\,I\,(\mathbf{x}_0,\mathbf{y}_0;\,\mathbf{x}_1,\ldots t_n)\qquad(3.18')$$

for the numerator, where $\mathbf{x}$, $\mathbf{y}$ are associated with the upper and lower ends of the interaction respectively. The integrand I is the product of the following factors:

(1) A factor

$$(2i)^{-1}v(\mathbf{x}_n-\mathbf{y}_n) \qquad (3.19)$$

for each interaction $n \neq 0$, and a factor $(2)^{-1}v(\mathbf{x}_0-\mathbf{y}_0)$ for the 0 interaction.

(2) A factor

$$G^0(x,x') = \langle 0\,|\,T(\psi(x),\psi^+(x'))\,|\,0\rangle \qquad (3.20)$$

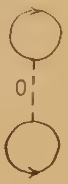
for a directed line from an upper end $x'(\equiv \mathbf{x}',t')$ to an upper end x, and similar factors

$$G^0(y,y'), \quad -G^0(x,y'), \quad -G^0(y,x') \qquad (3.21)$$

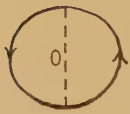
for lines from lower to lower ends, etc. Actually the minus signs here always cancel for the complete diagram.

(3) The directed lines form a number of closed loops, for each of which there is a factor (-1).

The lowest order numerator diagrams are

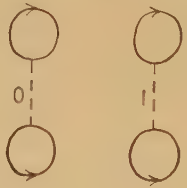


. . . . . (3.22)

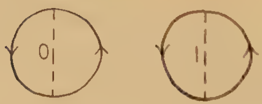


. . . . . (3.23)

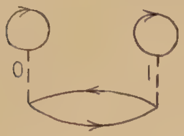
In the next order there are 4 ! = 24 diagrams of which a few are



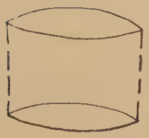
. . . . . (3.24)



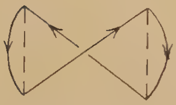
. . . . . . (3.25)



. . . . . . (3.26)



. . . . . . (3.27)



. . . . . . (3.28)

The diagrams (3.24) and (3.25) illustrate that the general numerator diagram consists of a part connected to the interaction line 0, and one or more separate parts. Now the sum of numerator diagrams containing such disconnected parts cancels exactly against the denominator in (3.8). For consider the general numerator diagram with interactions 0, 1, . . . N. Denote by α the part containing the interaction 0 and $n(\alpha)$ others; let I_α be the integrand which would be appropriate in the absence of other parts. Similarly let I_β be the integrand for the remaining part β, considered separately as if it were a denominator diagram of order $n(\beta)$. We need consider explicitly only those diagrams in which α involves the particular interactions 1, 2, 3, . . . $n(\alpha)$, multiplying by a factor $(N!/n(\alpha)!\,n(\beta)!)$ to allow for the other possibilities. Then the general numerator contribution has the form

$$\frac{N!}{n(\alpha)!\,n(\beta)!}\,\frac{(-i)^N}{N!}\int d\mathbf{x}_0\,d\mathbf{y}_0\,d\mathbf{x}_1\,d\mathbf{y}_1\,dt_1\ldots I_\alpha(x_0,y_0\,;x_1,y_1,t_1\,;\ldots)$$
$$\times\int d\mathbf{x}_1\,d\mathbf{y}_1\,dt_1\ldots I_\beta\,(x_1,y_1,t_1\,;\ldots).$$

Summing over all possible diagrams (α,β) the total numerator is

$$\sum_{\alpha,\beta}\frac{(-i)^{n(\alpha)+n(\beta)}}{n(\alpha)!\,n(\beta)!}\int d\mathbf{x}_0\,d\mathbf{y}_0\ldots I_\alpha\int d\mathbf{x}_1\,d\mathbf{y}_1\,dt_1\ldots I_\beta$$
$$=\left\{\sum_\alpha\frac{(-i)^{n(\alpha)}}{n(\alpha)!}\int d\mathbf{x}_0\,d\mathbf{y}_0\ldots I_\alpha\right\}\left\{\sum_\beta\frac{(-i)^{n(\beta)}}{n(\beta)!}\int d\mathbf{x}_1\,d\mathbf{y}_1\,dt_1\ldots I_\beta\right\}.$$

The second factor is just the expansion of the denominator, so we have the stated result. The quotient (3.8) is then the sum of numerator diagrams all of which are linked to the 0 interaction. Symbolically

$$\Delta E=\langle 0\,|\sum\frac{(-i)^n}{n!}\int_{-\infty}^{0}dt_1\ldots dt_n\,T(V(0)\,V(t_1)\ldots V(t_n))|\,0\rangle_{\mathrm{L}} \qquad (3.29)$$

where L means that only linked diagrams are included.

The propagator G^0 is explicitly

$$G^0(x,x') = \langle 0 \,|\, T(\psi(x), \psi^+(x')) \,|\, 0\rangle$$
$$= \Big\{\theta(t-t') \sum_{|\mathbf{k}|>k_f} - \theta(t'-t) \sum_{|\mathbf{k}|<k_f}\Big\} u_{\mathbf{k}}(\mathbf{x})\, u_{\mathbf{k}}^*(\mathbf{x}') \exp[-i(t-t')k^2/2M]$$
$$= \frac{1}{\Omega}\Big\{\theta(t-t') \sum_{|\mathbf{k}|>k_f} - \theta(t'-t) \sum_{|\mathbf{k}|<k_f}\Big\} \exp[i\mathbf{k}(\mathbf{x}-\mathbf{x}') - i(t-t')k^2/2M] \quad . \quad . \quad . \quad (3.30)$$

or, in the limit $\Omega \to \infty$,

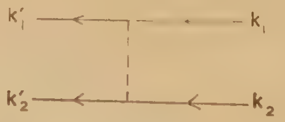
$$G^0(x,x') = (2\pi)^{-3}\Big\{\theta(t-t')\int_{k>k_f} d\mathbf{k} - \theta(t'-t)\int_{k<k_f} d\mathbf{k}\Big\}$$
$$\times \exp[i\mathbf{k}(\mathbf{x}-\mathbf{x}') - i(t-t')k^2/2M]. \quad . \quad . \quad . \quad . \quad . \quad (3.31)$$

With respect to spin indices G^0 is the unit matrix.

Provided that G^0 falls off sufficiently rapidly for large differences in its arguments, it is clear that for a linked diagram the integrations over $\mathbf{x}_1, \mathbf{y}_1, t_1$ etc. converge to a definite value for $\Omega \to \infty$ (we assume here that v has a finite range). The integration over $\mathbf{x}_0$ and $\mathbf{y}_0$ then gives a single factor Ω, so that the energy shift per unit volume approaches a constant. Thus every term in the linked diagram expansion for ΔE behaves reasonably. A diagram with n disconnected parts would give an integral proportional to Ω^n. It is the explicit cancellation of such terms at the outset that constitutes the solution of the problem stated in § 3.2.

From a given linked diagram distinct but equal contributions can be obtained (a) by permuting the labels $1 \dots n$ of interactions other than 0, and (b) by interchanging the roles of opposite ends ($x \rightleftharpoons y$) of one or more of these interactions. Only one member of such a class is ever considered explicitly; the remainder are allowed for simply by omitting (a) the factor $n!$ in (3.18) and (b) the factor ($\frac{1}{2}$) for each interaction other than 0.

Using (3.30) the space integrations in (3.18) can be replaced by sums over all values of labels $\mathbf{k}$, one attached to each directed line. An interaction such as



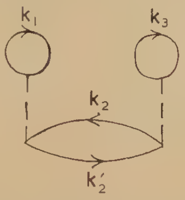
is then represented in the summand by a factor

$$(\mathbf{k}_1{}'\,\mathbf{k}_2{}' \,|\, v \,|\, \mathbf{k}_1\,\mathbf{k}_2) = \int d\mathbf{k}\, d\mathbf{y}\, u_{\mathbf{k}_1{}'}^*\, u_{\mathbf{k}_2{}'}^*\, v(\mathbf{x}-\mathbf{y})\, u_{\mathbf{k}_1} u_{\mathbf{k}_2}. \qquad (3.32)$$

This must contain a momentum conservation factor

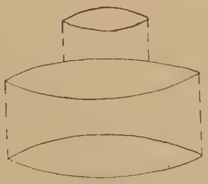
$$\delta_{(\mathbf{k}_1{}'+\mathbf{k}_2{}'),\,(\mathbf{k}_1+\mathbf{k}_2)}.$$

Because of this many diagrams that can be drawn do not in fact contribute. For example the diagram (3.26)

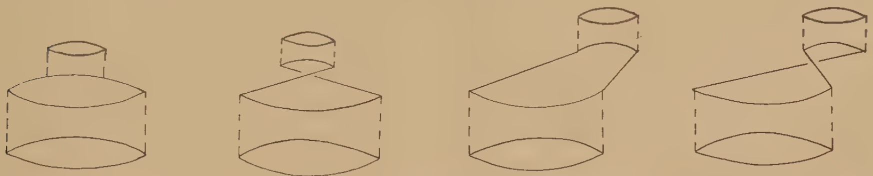


could contribute only when $\mathbf{k}_2$ and $\mathbf{k}_2'$ correspond to equal and opposite momentum. On the other hand, it follows from (3.30) that it could contribute only with $\mathbf{k}_2$ and $\mathbf{k}_2'$ on opposite sides of the fermi surface. There is therefore no contribution. In general we do not draw or mention such diagrams further.

When (3.30) is used the time integrations also can be performed. A diagram with $(n+1)$ interactions gives $n!$ distinct contributions, one for each possible order of the times $t_1 \ldots t_n$. It is convenient to represent these parts separately by *time-ordered* diagrams, as distinct from the original *Feynman* diagrams. In the new diagrams the order of the interactions becomes significant; those with earlier time argument standing further to the right. The associated integrals run over all positive values of the intervals between successive time arguments. For example the Feynman diagram



breaks up into the four time-ordered diagrams



and two more which vanish by momentum conservation. It is clear from (3.30) that a line directed from right to left must be associated with a state $\mathbf{l}$ above the fermi surface and a line from left to right with a state $\mathbf{m}$ below. Let $\sum(l^2/2M)$ be the sum of kinetic energies of lines crossing from right to left an interval between successive interactions and let $\sum(m^2/2M)$ be the corresponding sum for lines crossing the other way. Then each such interval contributes to the time integration a factor

$$\int_{-\infty}^{0} dt \exp\left[i\left(\sum l^2/2M - \sum m^2/2M\right)\right] = \frac{i}{\sum m^2/2M - \sum l^2/2M}. \tag{3.33}$$

This time integrated form is usefully compared with the result of ordinary time independent perturbation theory. The Brillouin–Wigner series for the energy shift is

$$\Delta E = \langle 0 | V + V \frac{Q}{\Delta E + T_0 - H_0} V + V \frac{Q}{\Delta E + T_0 - H_0} V \frac{Q}{\Delta E + T_0 - H_0} V + \ldots | 0 \rangle \qquad (3.34)$$

where, as before, T_0 is the unperturbed ground state energy. The projection operator Q excludes the unperturbed ground state. When expanded in terms of unperturbed intermediate states the series (3.34) can be represented by diagrams like the time-ordered diagrams introduced above; the forward and backward lines represent the particles and holes in the virtual excited states. To obtain from this the linked diagram expansion it is necessary to make the following changes (which compensate one another exactly):

(1) The quantity ΔE is omitted in energy denominators.

(2) Terms represented by diagrams with parts not connected to the extreme left interaction are omitted.

(3) The matrix elements of V are computed ignoring the presence of any particles or holes other than those actually scattered, created, or destroyed, by the particular interaction involved.

As a result of (3) there are non-vanishing contributions from intermediate states in which several particles appear to occupy the same state above the fermi surface, or several holes the same state below†. Thus the lines in the diagrams for the linked cluster expansion strictly do not denote occupied states or holes, but only operator pairings in the application of Wick's theorem.

Implying both (2) and (3) in the subscript L we write symbolically

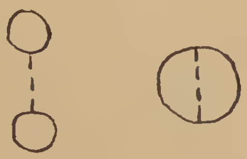
$$\Delta E = \left\{ \langle 0 | V + V \frac{Q}{T_0 - H_0} V + \ldots | 0 \rangle \right\}_{\mathrm{L}} .$$

The existence of such a linked diagram expansion was conjectured by Brueckner (1955 a, b). It was established by Goldstone (1957) essentially as above. The introduction of time is not of course essential in a stationary state problem. It is avoided in the much more complicated work of Hugenholtz (1957 a). Several other papers also claim to establish linked diagram expansions in a time independent way (e.g. Riesenfeld and Watson 1956, Coester 1958).

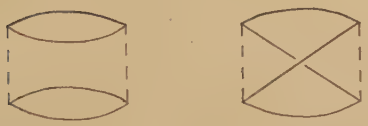
---

† This point is much emphasized by Katz (1960 a, b).

Of course the method of this section reproduces the low order perturbation formulae used in § 1. In first and second order the only contributions are



(3.35)



(3.35′)

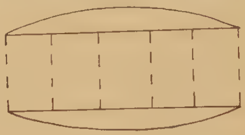
These lead at once to (1.19) and (1.33), the second member of each pair being the exchange contribution.

3.4. *The Low Density Limit*

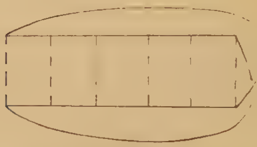
As has been observed in § 1 individual terms in perturbation theory are not sensible with hard core potentials, but reasonable results can perhaps be obtained by summing an infinite subset. The subset proposed by Brueckner can be given a formal basis (following Hugenholtz 1957 b) in the low density limit. Now nuclear matter is not a low density system, the overall range of the forces being comparable with the interparticle spacing. However, in so far as the inclusion of higher order terms is dictated mainly by the very strong short-range part of the potential, in the selection of such terms low density considerations are relevant.

According to Hugenholtz the dominant time-ordered diagrams at low density are those with the least number of backward, or hole, lines. He argues that each such line is associated with an integration over momenta below the fermi surface and so introduces at low density a factor k_f^3. Actually momentum conservation often cuts down the number of such integrations which are independent, and clearly only independent integrations below k_f should be counted. The argument would then be sufficient if the remaining integrations were over a finite domain and if the integrand were bounded independently of k_f. The first condition may be effectively secured by the fall-off at high momenta of the interaction matrix elements—always supposing for formal arguments that the hard core is very large but not actually singular. However, the second condition does not hold; the integrand is unbounded because energy denominates can be arbitrarily small. Moreover, in at least one case the formal sum of the series has a low density behaviour different from that attributed by this argument to the individual terms (§ 3.5). Nevertheless, for want of better arguments, we proceed on the assumption that a first approximation need include only those terms with the minimum of independent hole lines.

Every diagram has at least two independent hole lines. Those with only two are of the following types, with any number of interaction lines:

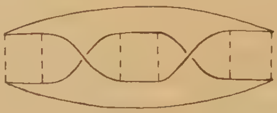


. . . . (3.36)

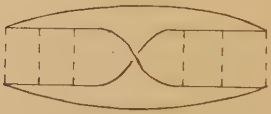


. . . . (3.37)

Note again that diagrams differing from these only by the turning over of a number of interaction lines, such as



or



are automatically allowed for by the modified rules of calculation on page 244. With one interaction line only we obtain from the two diagrams the first order energy shift

$$\tfrac{1}{2}\sum_{\mathbf{m}\,\mathbf{m}'}\langle\mathbf{mm}'|v|\mathbf{mm}'-\mathbf{m}'\mathbf{m}\rangle. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.38)$$

It is readily confirmed that all higher order 'ladder' diagrams (3.36) and (3.37) are included by replacing the matrix elements of v by those of $K(m^2+m'^2/2M)$ where K is determined by

$$K(\mathscr{E})=v+\sum_{\mathbf{l}\,\mathbf{l}'}\frac{v|\mathbf{ll}'\rangle\langle\mathbf{ll}'|K(\mathscr{E})}{\mathscr{E}-(l^2+l'^2)/2M} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.39)$$

or, in more compact operator notation,

$$K(\mathscr{E})=v+v\frac{Q}{\mathscr{E}-\mathscr{H}_0}K(\mathscr{E}). \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.39')$$

Here $\mathscr{H}_0$ is the two-body kinetic energy operator, and Q is a projection operator onto states above the fermi surface. Solution of this equation by iteration:

$$K(\mathscr{E})=v+v\frac{Q}{\mathscr{E}-\mathscr{H}_0}v+v\frac{Q}{\mathscr{E}-\mathscr{H}_0}v\frac{Q}{\mathscr{E}-\mathscr{H}_0}v+\ldots \quad . \quad . \quad (3.40)$$

provides explicitly the contributions of individual higher order diagrams. Equation (3.39), together with

$$\Delta E = \tfrac{1}{2} \sum_{\mathbf{m}\,\mathbf{m}'} \langle \mathbf{m}\mathbf{m}' \,|\, K\left(\frac{m^2+m'^2}{2M}\right) |\, \mathbf{m}\mathbf{m}' - \mathbf{m}'\mathbf{m} \rangle, \quad . \quad . \quad (3.41)$$

defines the simplest form of Brueckner theory. It is obviously correct to second order in perturbation theory and contains all those terms which appear to be important for a strong core of sufficiently short range. In the following section we discuss methods for solving (3.40) and applications thereof to nuclear matter.

3.5. *The K-matrix Equation*

The expression (3.41) for the energy shift involves directly only the diagonal and exchange matrix elements:

$$\langle \mathbf{m}\mathbf{m}' \,|\, K\left(\frac{m^2+m'^2}{2M}\right) |\, \mathbf{m}\mathbf{m}' \rangle \quad \text{and} \quad \langle \mathbf{m}\mathbf{m}' \,|\, K\left(\frac{m^2+m'^2}{2M}\right) |\, \mathbf{m}'\mathbf{m} \rangle.$$

These are 'on the energy shell', in that the argument $\mathscr{E}$ of $K(\mathscr{E})$ equals the total kinetic energy of both initial and final states. We consider first an approximation in which such matrix elements are expressed directly in terms of scattering phase shifts without reference to the potential.

We have to solve

$$K(\mathscr{E}) = v + L(\mathscr{E})\, K(\mathscr{E}) \quad . \quad . \quad . \quad . \quad . \quad (3.42)$$

where

$$L(\mathscr{E}) = \sum_{\mathbf{l}\,\mathbf{l}'} \frac{|\mathbf{l}\mathbf{l}'\rangle\langle \mathbf{l}\mathbf{l}'|}{\mathscr{E} - (l^2 + l'^2)/2M}. \quad . \quad . \quad . \quad (3.43)$$

For comparison, the free space reaction matrix K^0 is given by

$$K^0(\mathscr{E}) = v + vL^0(\mathscr{E})\, K^0(\mathscr{E}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.44)$$

where

$$L^0(\mathscr{E}) = P \sum_{\mathbf{k}\,\mathbf{k}'} \frac{|\mathbf{k}\mathbf{k}'\rangle\langle \mathbf{k}\mathbf{k}'|}{\mathscr{E} - (k^2 + k'^2)/2M}. \quad . \quad . \quad . \quad . \quad (3.45)$$

Here the intermediate states are not restricted to being below the fermi surface. For the interesting values of $\mathscr{E}$ the denominator can then vanish, and the principal part of the singular integral is taken, as indicated by the P in (3.45). Now K can be expressed in terms of K^0, for a formal solution of (3.42) is

$$K = K^0 + K^0 (L - L^0)\, K. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.46)$$

If the free space scattering K^0 is weak the second term can be ignored. For spinless particles then

$$\langle \mathbf{m}\mathbf{m}' \,|\, K((m^2+m'^2)/2M) \,|\, \mathbf{m}\mathbf{m}' \rangle = \Omega^{-1} \frac{4\pi}{Mk} \sum_l (2l+1) \tan \delta_l(k) \quad . \quad . \quad . \quad (3.47)$$

where $\delta_l(k)$ is the scattering phase shift for the lth partial wave and k is the relative momentum

$$k = \tfrac{1}{2}\,|\mathbf{m} - \mathbf{m}'|.$$

Similarly

$$\langle \mathbf{m}\mathbf{m}'\,|K((m^2+m'^2)/2M)|\,\mathbf{m}'\mathbf{m}\rangle = \Omega^{-1}\frac{4\pi}{Mk}\sum_l (2l+1)\,(-1)^l \tan\delta_l(k). \quad . \quad . \quad . \quad (3.48)$$

For nucleons it is convenient to work, not with the independent spin states of two particles, but with the singlet–singlet, singlet–triplet, triplet–singlet, and triplet–triplet combinations. In the summation (3.41) these take the appropriate statistical weights 1, 3, 3, and 9 respectively. For each combination, assuming central forces only, formulae (3.47) and (3.48) hold with appropriate phase shifts, except that (3.48) changes sign when the overall spin state is odd under exchange. The energy shift is then

$$\Delta E = \frac{2\pi}{M\Omega}\sum_{\mathbf{m}\,\mathbf{m}'}\left\{\frac{2}{k}\sum_{l\ \text{even}}(2l+1)\,(3\tan\delta_l^{13}(k)+3\tan\delta_l^{31}(k)) + \frac{2}{k}\sum_{l\ \text{odd}}(2l+1)\,(\tan\delta_l^{12}(k)+9\tan\delta_l^{33}(k))\right\}. \quad . \quad . \quad . \quad . \quad (3.49)$$

With

$$A = 4\sum_{\mathbf{m}} = \frac{4\Omega}{(2\pi)^3}\int d\mathbf{m}$$

in the limit $\Omega \to \infty$, and the integration over $(\mathbf{m}+\mathbf{m}')$ carried out, the energy shift per particle is

$$\frac{\Delta E}{A} = \frac{4k_f^2}{\pi M}\int_0^{k_f}\frac{k\,dk}{k_f^2}\left\{1-\tfrac{3}{2}\frac{k}{k_f}+\tfrac{1}{2}\left(\frac{k}{k_f}\right)^3\right\} \times \left\{\sum_{l\ \text{even}}(2l+1)(3\tan\delta_l^{13}+3\tan\delta_l^{32}) + \sum_{l\ \text{odd}}(2l+1)(\tan\delta_l^{11}+9\tan\delta_l^{33})\right\}. \quad . \quad . \quad . \quad (3.50)$$

For nuclear matter at normal density the phase shifts needed are those up to a laboratory energy of about 160 Mev, the most heavily weighted being around 50 Mev.

Formula (3.50) and a generalized version permitting tensor forces were used in the early work of Brueckner *et al.* (1954) and Brueckner (1954). The phase shifts were computed from a potential determined in part by meson theory and with a phenomenological hard core. Reasonable values were obtained for the energy and the mean spacing when the tensor force was omitted, but its inclusion resulted in both quantities becoming much too small.

In such calculations there occurs the difficulty that, because of the bound deuteron state, the triplet s phase shift approaches π rather than 0 at zero energy, and passes through $(\pi/2)$ in the range of integration of (3.50). The integrand is then singular, and Brueckner *et al.* took the principal value of the integral. However, in such circumstances the formula is just not applicable. The derivation implied weak scattering and so small phase shifts. Indeed we are entitled to make the substitution

$$\tan\delta \rightarrow \delta \qquad (3.51)$$

which at least removes the singularity if the formula is used for large phase shifts. Now it happens that (in the absence of bound states) such a formula then gives the exact answer for ΔE in the trivial 'many-body problem with one-body forces' (Bell 1959, Katz 1960 a, b). Moreover, the second virial coefficient for a non-degenerate quantum gas with two-body forces is also linear in the phase shifts (Beth and Uhlenbeck 1936), with again extra bound state terms. So attempts have been made to provide in the present case also some real reason for preferring δ to $\tan\delta$ (Fukuda and Newton 1956, de Witt 1956). That these are futile is indicated for example by the case, now to be considered, of the pure hard core interaction. There either δ or $\tan\delta$ gives correctly the term of first order in the core radius, but neither predicts the presence of the second-order term.

For the pure hard core the s-wave phases are simply kc, where c is the core radius. So from (3.50) to first order in c

$$\left(\frac{\Delta E}{A}\right) = \left(\frac{k_f^2}{M}\right)\left(\frac{k_f c}{\pi}\right), \qquad (3.52)$$

a formula due to Lenz (1929). Consider now further terms in a systematic expansion in powers of $(k_f c)$. The expansion of $\tan\delta$ for s-waves gives a contribution of order $(k_f c)^3$, as do higher partial waves. Diagrams other than the ladders (3.36) and (3.37) are of third and higher order in perturbation theory, and so also of at least this order in K^0, or c, in a rearranged series†. The only contribution of order c^2 is then from the second term in (3.46) which is adequately approximated by

$$K^0 (L - L^0) K^0. \qquad (3.53)$$

To compute this, off-diagonal matrix elements of K^0 are required. It is readily shown (by methods such as those used later in this section) that to lowest order in c

$$\langle \mathbf{m}_1 \mathbf{m}_2 \,|\, K^0(\mathscr{E}) \,|\, \mathbf{m}_1{}' \, \mathbf{m}_2{}' \rangle = \Omega^{-1}\, \delta\, (\mathbf{m}_1 + \mathbf{m}_2), (\mathbf{m}_1{}' + \mathbf{m}_2{}') \left(\frac{4\pi c}{M}\right). \qquad (3.54)$$

Using this in (3.53) and (3.41) and performing intermediate state integrations (Levinger *et al.* 1960) yields the correction to (3.52)

$$+ \left(\frac{k_f^2}{M}\right) \frac{6}{35\pi^2} (11 - 2 \ln 2)\, (k_f c)^2. \qquad (3.55)$$

---

† On dimensional grounds they then contribute as $(k_f c)^3$, which in terms of k_f is of order k_f^2 relative to the leading term. On the other hand, such diagrams have one more hole line at least, and on this ground might have been thought to be of relative order k_f^3. This is the point mentioned at the beginning of § 3.4.

This result was obtained in essentially this way by Martin and de Dominicis (1957) and Galitskii (1958), and in different ways by Huang and Yang (1957) and Yang and Lee (1957). Martin and de Dominicis gave also an approximate value for the next term. Including this, we have

$$\frac{\Delta E}{A}=\frac{k_f^2}{2M}\{0{\cdot}63\,(k_fc)+0{\cdot}32\,(k_fc)^2+0{\cdot}78\,(k_fc)^3+\dots\}. \qquad (3.56)$$

For nuclear matter, with $k_f=1{\cdot}5f^{-1}$ and $c=0{\cdot}4f^{-1}$,

$$\Delta E/A=40\{0{\cdot}35+0{\cdot}10+0{\cdot}14+\dots\}\,\text{Mev}.$$

Clearly the core radius is not small enough for convincing convergence. The terms given add to about 24 Mev, about the same as the mean unperturbed kinetic energy.

If the core radius c were sufficiently small and the remainder of the potential sufficiently weak, the latter could be allowed for simply by adding its expectation value in the unperturbed state to (3.56). This approximation was used for nuclear matter by Gomes *et al.* (1958). Their potential, fitted to low energy singlet scattering, was a hard core plus square well attraction. The whole potential, including the core, had a Serber exchange character. The coefficient of the third order term in (3.56) with no p-wave contributions and no contributions from diagrams other than the basic ladder diagrams (3.36) and (3.37), was found to be 0·26 rather than 0·78. The equilibrium spacing came out very close to the empirical value and the binding energy per particle was about 7 Mev. This result was thought reasonable in view of the omission of the tensor force.

The validity of this approximation was examined by Levinger *et al.* (1960 a, b). Let v_S denote the hard core and v_1 the remainder of the potential, so that $v=v_S+v_1$ (this corresponds to taking $d=c$ in (2.24)). If K_S is the solution of (3.42) with v_S alone, K can be developed in powers of K_S and v_1. As far as second-order terms:

$$K=K_S+v_1+v_1\,L\,v_1+K_S\,L\,v_1+v_1\,L\,K_S. \quad . \quad . \quad . \quad . \qquad (3.57)$$

The first two terms are those retained by Gomes *et al.* (1958). To estimate the corrections K_S is replaced by the first approximation (3.54). The third term is just the second-order energy due to a regular potential and might be expected to be small from the work of § 1.4. However, the presence of the hard core requires v_1 to be stronger than would otherwise be necessary to fit low energy scattering. As a result the second order term in v_1 is quite large, contributing about (-20) Mev per particle to the energy. The remaining terms in (3.57) express the fact that the hard core reduces the wave function in its vicinity and so partially suppresses the effect of v_1. Levinger *et al.* find from these terms an energy of about $(+20)$ Mev per particle. There is considerable numerical cancellation among the correction terms, but they are individually so large that the convergence of the method is very doubtful.

Levinger *et al.* used for v_1 conventional forms in which the strongest part is also the innermost, and so most affected by the proximity of the core.

With the square wells of Gomes *et al.* the correction should be smaller; The numerical examination of this particular point by Moszkowski and Scott (1960) is vitiated by their omission to vary the effective mass self consistently with density.

The method of Moszkowski and Scott (1960) exploits the substantial cancellation between the effects of the hard core and the innermost part of the attractive potential. By allowing v_s to include in addition to the hard core some part of the attraction, out say to a definite radius (see 2.24)) it is possible to weaken both K_s^0 and v_1 and so reduce the second-order terms. One choice of d is such that v_s gives no free space scattering at zero energy. Not much appears to be gained by letting d vary with momentum and angular momentum. Moszkowski and Scott estimate the second-order corrections† for a Serber potential, with 0·4f hard core and an exponential attractive well, fitted to low energy singlet data. The last two terms in (3.57), the cross terms, are the largest; they contribute about 2 MeV per particle at normal density. The other terms are less by an order of magnitude. Thus the method is much more acceptable than that of Gomes *et al.* With the potential mentioned no equilibrium is found near normal density. However, by omitting the substantial d-wave and small g-wave contributions (i.e. by retaining interaction in s-states only) reasonable equilibrium density and energy are found‡. It is argued that this actually simulates the effect of the tensor force. But clearly it is desirable to extend the method to handle the tensor force properly, and also the large self-consistent field effect found by Brueckner and Gammel (§ 3.7). This has now been done (Moszkowski and Scott, private communication) §. Essentially, the results of Brueckner and Gammel are reproduced with much less labour. Details are not yet available.

Finally we outline the direct numerical solution of the K matrix equation by partial wave analysis. The spin states are dealt with in the singlet–singlet, singlet–triplet, etc. combinations; a separate equation holds for each, involving the relevant part of the potential. Because of conservation of momentum, the total momentum $\mathbf{p}$ of the two particles is a fixed parameter. We write

$$\langle \tfrac{1}{2}\mathbf{p}+\mathbf{k}', \tfrac{1}{2}\mathbf{p}-\mathbf{k}' | K(\mathscr{E}) | \tfrac{1}{2}\mathbf{p}+\mathbf{k}, \tfrac{1}{2}\mathbf{p}-\mathbf{k} \rangle = \langle \mathbf{k}' | K(\mathscr{E},\mathbf{p}) | \mathbf{k} \rangle \qquad (3.58)$$

where
$$\langle \mathbf{k}' | K(\mathscr{E},\mathbf{p}) | \mathbf{k} \rangle = \langle \mathbf{k}' | v | \mathbf{k} \rangle + \sum_{\mathbf{k}} \frac{\langle \mathbf{k}' | v | \mathbf{k}'' \rangle f(\mathbf{k}'',\mathbf{p}) \langle \mathbf{k}'' | K(\mathscr{E},\mathbf{p}) | \mathbf{k} \rangle}{\mathscr{E} - \dfrac{1}{2M} |\tfrac{1}{2}\mathbf{p}+\mathbf{k}''|^2 - \dfrac{1}{2M} |\tfrac{1}{2}\mathbf{p}-\mathbf{k}''|^2}, \quad . \; . \; . \quad (3.59)$$

---

† In these estimates some allowance is made for self-consistent modification (see § 3.6 and § 3.7) of the energy denominators, but this should not alter the order of magnitude.

‡ In these calculations the third order self-consistent field effect of § 3.6 is included, and plays an important role in locating the equilibrium.

§ See *Bull. Amer. phys. Soc.*, **6**, 149 (1961).

and f expresses the exclusion principle effect

$$\begin{aligned} f(\mathbf{k}'', \mathbf{p}) &= 1 \quad \text{if} \quad |\tfrac{1}{2}\mathbf{p} \pm \mathbf{k}''| > k_f \\ &= 0 \quad \text{otherwise.} \end{aligned} \qquad (3.60)$$

Clearly

$$\langle \mathbf{k}' | K(\mathscr{E}, \mathbf{p}) | \mathbf{k} \rangle = \int d\mathbf{r} \exp(i\mathbf{k}' . \mathbf{r})\, v(r)\, \psi_{\mathbf{k}}(\mathbf{r}) \qquad (3.61)$$

where $\psi_{\mathbf{k}}(\mathbf{r})$, dependent of course also on $\mathscr{E}$ and $\mathbf{p}$, is determined by

$$\psi_{\mathbf{k}}(\mathbf{r}) = \exp(i\mathbf{k} . \mathbf{r}) + \int d\mathbf{r}'\, G(\mathbf{r}, \mathbf{r}')\, v(\mathbf{r}')\, \psi_{\mathbf{k}}(\mathbf{r}') \qquad (3.62)$$

with

$$G(\mathbf{r}, \mathbf{r}') = \sum_{\mathbf{k}} f(\mathbf{k}, \mathbf{p}) \exp[i\mathbf{k} . (\mathbf{r} - \mathbf{r}')] \left\{ \mathscr{E} - \frac{p^2}{4M} - \frac{k^2}{M} \right\}^{-1}. \qquad (3.63)$$

Following Brueckner and Gammel (1958) we work directly with the integral eqn. (3.62). Actually the first general investigation of the K-matrix, made by Bethe and Goldstone (1957) used an integro-differential equation obtained by operating on (3.62) with

$$\left(\frac{1}{4M} \frac{\partial^2}{\partial r^2} + \mathscr{E} - \frac{p^2}{4M} \right).$$

This gives the inhomogeneous *Bethe–Goldstone equation*

$$\left(\mathscr{E} - \frac{p^2}{4M} + \frac{1}{M} \frac{\partial^2}{\partial r^2} \right) \psi_{\mathbf{k}}(r) = \left(\mathscr{E} - \frac{p^2}{4M} + \frac{k^2}{M} \right) \exp(i\, \mathbf{k} . \mathbf{r}) + \int d\mathbf{r}'\, g(\mathbf{r}, \mathbf{r}')\, v(r')\, \psi_{\mathbf{k}}(\mathbf{r}') \qquad (3.64)$$

with

$$g(\mathbf{r}, \mathbf{r}') = \sum_{\mathbf{k}} f(\mathbf{k}, \mathbf{p}) \exp[i\mathbf{k} . (\mathbf{r} - \mathbf{r}')]. \qquad (3.65)$$

The equation becomes homogeneous on the energy shell, $\mathscr{E} = (p^2/4M + k^2/M)$

Equation (3.62) differs from the free space scattering equation only in that G contains the exclusion factor f. This has two important effects. The first is that the energy denominator is never zero (for values of $\mathscr{E}$ needed in (3.41)), and the scattered wave therefore decreases more rapidly than r^{-1}. This is important when $\psi_{\mathbf{k}}$ is considered as a Jastrow correlation factor (§ 2.1); it is of no special importance here, as is clear from (3.61). The second effect is that for $\mathbf{p} \neq 0$ the Green function G is anisotropic. In a partial wave analysis the various waves are then coupled even for a central potential. The general coupled wave analysis is set out by Werner (1959). Brueckner and Gammel remove the difficulty by approximating f by its average $\bar{f}$ over directions of $\mathbf{p}$. The quantity f is unity if

$$|\cos\theta| < (kp)^{-1} (\tfrac{1}{4}p^2 + k^2 - k_f^2) \qquad (3.66)$$

where θ is the angle between $\mathbf{k}$ and $\mathbf{p}$, and zero otherwise. Averaging over angles we obtain

$$\begin{aligned} \bar{f}(p, k) &= 0, \quad \tfrac{1}{4}p^2 + k^2 < k_f^2 \\ &= 1, \quad |\tfrac{1}{2}p - k| > k_f \\ &= (kp)^{-1} (\tfrac{1}{4}p^2 + k^2 - k_f^2) \quad \text{otherwise.} \end{aligned} \qquad (3.67)$$

When ψ is then resolved into partial waves (characterized by quantum numbers L, S, J, J_z, T_3) the only remaining coupling is that due to the tensor force between triplet states with $L = J \pm 1$. The radical functions for these states are determined by pairs of integral equations, those for other states satisfying separate single equations. The details of this, given by Brueckner and Gammel, raise no question of principle. The only other point we consider is the special treatment demanded by the hard core.

To illustrate this we consider for simplicity only the singlet s-wave. The s-wave part of $\exp(i\mathbf{k}\,.\,\mathbf{r})$ is

$$\tfrac{1}{2}\int_{-1}^{+1} d\mu \exp(ikr\mu) = \frac{\sin kr}{kr} = r^{-1}\phi_0(\mathbf{r}). \quad . \quad . \quad . \quad (3.68)$$

From (3.62) the appropriate integral equation is

$$\phi(r) = \phi_0(r) + \int_0^\infty G(r, r')\, v(r')\, \phi(r')\, dr' \quad . \quad . \quad . \quad . \quad (3.69)$$

with

$$G(r, r') = \pi r r' \int\!\!\int d\mu\, d\mu' \sum_{\mathbf{k}} \frac{\bar{f}(k, p) \exp ik(r\mu - r'\mu')}{\mathscr{E} - \dfrac{p^2}{4M} - \dfrac{k^2}{M}}$$

$$= \frac{2}{\pi}\int dk \frac{\bar{f}(k, p) \sin kr \sin kr'}{\mathscr{E} - \dfrac{p^2}{4M} - \dfrac{k^2}{M}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.70)$$

Now with $r < c$, the potential is infinite. From (3.69) ϕ must then be zero, and the otherwise indeterminate combination $v\phi$ must be chosen to bring this about. To simplify the problem note now that the inside of the hard core has no physical significance, and can be removed if we wish without changing physical results. An extreme case of such a 'hollow' hard core is a δ-function *of infinite strength* at $r = c$. The contribution of this to $v\phi$ is of the form $\lambda\delta(r - c)$, where λ is a constant to be determined (it represents the product of the infinite strength of the potential and the zero wave function). Then (3.69) becomes

$$\phi(r) = \phi_0(r) + \lambda G(r, c) + \int_c^\infty dr'\, G(rr')\, v(r')\, \phi(r'), \quad . \quad . \quad (3.71)$$

which together with

$$\phi(c) = 0 = \phi_0(c) + \lambda G(c, c) + \int_c^\infty dr'\, G(c, r')\, v(r')\, \phi(r'), \qquad (3.72)$$

gives

$$\phi(r) = \left\{\phi_0(r) - \phi_0(c)\frac{G(r, c)}{G(c, c)}\right\}$$

$$+ \int_c^\infty dr' \left\{G(r, r') - \frac{G(r, c)\, G(c, r)'}{G(c, c)}\right\} v(r')\, \phi(r'). \quad . \quad . \quad (3.73)$$

The hard core is thus eliminated at the expense of complicating the Green function and the inhomogeneous term. This equation and corresponding ones for other waves are suitable for computation. The K-matrix contributions are finally obtained as integrals of the type

$$\begin{aligned}\int dr\,\phi_0(r)\,v(r)\,\phi(r) &= \lambda\phi_0(c)+\int_c^\infty dr\,\phi_0(r)\,v(r)\,\phi(r)\\ &= -\frac{\phi_0^2(c)}{G(c,c)}+\int_c^\infty dr\left\{\phi_0(r)-\phi_0(c)\,\frac{G(c,r)}{G(c,c)}\right\}v(r)\,\phi(r). \quad (3.74)\end{aligned}$$

Now this approach gives an accurate solution for what may be called a 'thin hollow hard core'. It is also an approximate solution for a 'solid' hard core, but there corrections are needed because (ϕv) does not vanish for $r<c$. There are indications (Bethe and Goldstone 1957, Brueckner and Gammel 1958, Appendix B) that such corrections are negligible in nuclear matter. Any appreciable difference in results, for differing assumptions about the inside of the core, would indicate inadequacy of the approximate theory being discussed.

The method just described was applied by Brueckner and Gammel (1958) using the potential described in § 1.3. The Green functions were constructed and the integral equations solved numerically, with rather coarse lattices. Actually the work was performed merely as the first step in a much more elaborate programme (§ 3.8). The main point of interest is that the binding energy per particle from this first approximation exceeds by about 20 Mev (at normal density) the final result. So with this potential, the most realistic used in such calculations, the simple version of the theory so far developed is rather poor.

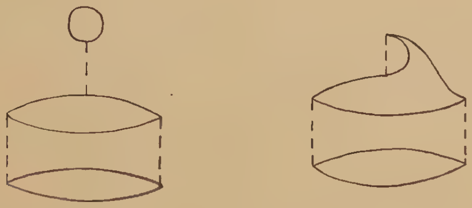
3.6. *Third-order Corrections*

We consider now corrections to the Brueckner ladder approximation. All first and second-order perturbation diagrams are already included, but most of the third-order diagrams are omitted. With singular potentials the perturbation theory terms are not immediately useful, and we go over to a modified expansion in K rather than v. The low order terms are most quickly generated simply by solving (3.42) for v in powers of K,

$$v=K(\mathscr{E})-K(\mathscr{E})\,L(\mathscr{E})\,K(\mathscr{E})+\dots \quad . \quad . \quad . \quad . \quad (3.75)$$

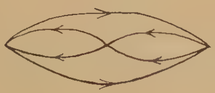
and substituting in the perturbation series. The first corrections to the theory of the last section are then of third order in K, and obtained merely by using matrix elements of K rather than v in the third-order perturbation diagrams. Note that any reasonable value of $\mathscr{E}$ is permissible in (3.75); the $\mathscr{E}$ dependence of K is clearly a second-order effect and is ignored in the crude estimates to be considered. The use of K rather than v is roughly equivalent to considering not individual third-order perturbation diagrams but whole families generated from them by replacing interaction lines by ladders. The cavalier treatment of the $\mathscr{E}$ dependence of K prevents such a correspondence being precise.

A particular class of third-order diagrams is obtained by the insertion, into one or more lines of second-order diagrams, of *self-energy parts.* Examples are

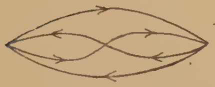


. . . (3.76)

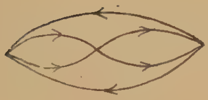
This class we set aside for the present. The remaining third-order diagrams are



'particle–particle' scattering (3.77)



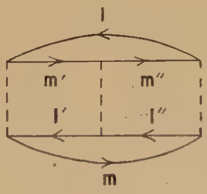
'particle–hole' scattering . . . (3.78)



'hole–hole' scattering . . . (3.79)

where, to avoid for the moment distinction between their ends, we have used interaction lines of zero length. The first of these is actually included already in the ladder sequence. The others are new.

The 'particle–hole scattering' is also known as the 'three-body cluster' term; this is because it has three hole lines and may be thought of as a collision of three particles from the fermi sea. It was estimated by Brueckner (1955 b), with extensive revision by Bethe (1956). The component diagram

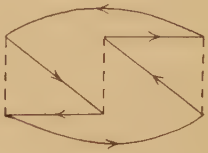


. . . . (3.80)

gives a contribution

$$-\frac{\langle \mathbf{m'm}\,|\,K\,|\,\mathbf{l'l}\rangle\langle \mathbf{m''l'}\,|\,K\,|\,\mathbf{m'l''}\rangle\langle \mathbf{ll''}\,|\,K\,|\,\mathbf{m''m}\rangle}{(m^2+m'^2-l^2-l'^2)(m^2+m''^2-l^2-l''^2)/4M^2}. \tag{3.81}$$

The factor ($\frac{1}{2}$) normally associated with the first interaction is omitted; this is compensated by a factor 2 arising from the alternative time ordering of the other interactions. Distinct diagrams are obtained by letting a line to the top of an interaction go instead to the bottom and vice versa, for example



Such variations can be allowed for by the substitution in (3.81)

$$K \rightarrow K(1 - P^r P^\sigma P^\tau)$$

where the P are exchange operators in space, spin, and isobaric spin. We must not consider also interchanges of lines going away from an interaction; this would involve counting separately diagrams differing only by interchanging the roles of opposite ends of interactions. As regards the second and third interactions this is already allowed for in the rules of calculation. As regards the first interaction it is equivalent (in the Feynman diagram) to interchanging the remaining interactions, also allowed for already.

Brueckner and Bethe approximate the matrix elements of K by those of a conventional Serber potential fitted to low energy singlet scattering. The approximation cannot be good; it does not for example reproduce the repulsive effect of the hard core which dominates for large momentum transfer. However, it should give an order of magnitude. With w just a function of internucleon distance then

$$K = \tfrac{1}{2}w(1 + P^r), \qquad (3.82)$$

$$K(1 - P^r P^\sigma P^\tau) = \tfrac{1}{2}w(1 + P^r)(1 - P^\sigma P^\tau). \qquad (3.83)$$

The spin summation implied in (3.81) involves the trace over the spin states of three particles:

Trace

$$(1 - P_{12}{}^\sigma P_{12}{}^\tau)(1 - P_{23}{}^\sigma P_{23}{}^\tau)(1 - P_{31}{}^\sigma P_{31}{}^\tau) = 12. \qquad (3.84)$$

This is calculated by using the expressions

$$P_{12}{}^\sigma = \tfrac{1}{2}(1 + \boldsymbol{\sigma}_1 . \boldsymbol{\sigma}_2), \text{ etc.,}$$

and reducing the expression to one linear in the spin operators of each particle, whose traces are zero. Brueckner (1955 a, b) quotes a spin factor 24. Bethe (1956) obtains the factor 12, but thought that he had omitted some terms (see his footnote); however, they are just exchange terms already allowed for. With the notation

$$\mathbf{l} = \mathbf{m} - \mathbf{q}, \quad \mathbf{l}' = \mathbf{m}' + \mathbf{q}, \quad \mathbf{l}'' = \mathbf{m}'' + \mathbf{q}$$

and converting sums to integrals, the total contribution to the energy shift per particle is

$$(\Delta E/A)_{\mathrm{pp}} = \frac{9M^2}{2^{14}\pi^{10}k_f^{\,3}}\int d\mathbf{q}\, d\mathbf{m}\, d\mathbf{m}'\, d\mathbf{m}''$$
$$\times \frac{\{w(q)+w(|\mathbf{q}+\mathbf{m}'-\mathbf{m}|)\}\{w(q)+w(|\mathbf{m}''-\mathbf{m}'|)\}\{w(q)+w(|\mathbf{q}+\mathbf{m}''-\mathbf{m}|)\}}{(q^2+\mathbf{q}\,.\,\mathbf{m}-\mathbf{q}\,.\,\mathbf{m}')(q^2+\mathbf{q}\,.\,\mathbf{m}-\mathbf{q}\,.\,\mathbf{m}'')} \quad . \quad . \quad . \quad (3.85)$$

where $w(q)$ is the Fourier transform

$$w(q) = \int d\mathbf{r} \exp(-i\mathbf{q}\,.\,\mathbf{r})\, w(r).$$

Bethe finds†, with an approximate evaluation of the integral,

$$(\Delta E/A)_{\mathrm{pp}} \approx -0{\cdot}3\ \text{MeV}$$

at normal density. Brueckner (1958) has a similar result.

With the particular approximation used to evaluate the integral, the correction is, at low density, proportional to k_f, and so more important relative to the leading term in the energy shift ($\propto k_f^{\,3}$). Bethe suggests that this indicates the breakdown of the Brueckner method and the formation of real clusters. However, the approximations used in the integration were suitable only for high density (Brueckner 1955 b). The result, not very reliable at normal density, therefore permits no conclusion about low density. According to Hugenholtz (§ 3.4) the cluster diagram would be relatively *less* important at very low density because it has an extra hole line. But this would not mean the improved convergence of the method as a whole, for clustering and indeed condensation must occur with realistic forces. These phenomena escape the essentially perturbative approach.

Bethe (1956) has made a second estimate, to some extent complementary with the first. In this the hard core alone is considered. In the short range limit (from (3.54))

$$w(q) = 4\pi c/M. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.86)$$

In this limit the assumption of Serber exchange character is unimportant ($P^r \equiv 1$) and we can substitute (3.86) in (3.85). Simplifying the integral by ignoring the m^2 in the denominators and integrating over all q greater than k_f we get the rough estimate

$$(\Delta E/A)_{\mathrm{pp}} \approx 0{\cdot}021(k_f c)^3\,(k_f^{\,2}/2M)$$
$$\approx +0{\cdot}2\ \text{MeV}$$

with $c = 0{\cdot}4f$, $k_f = 1{\cdot}5f^{-1}$.

Finally we quote a value for the cluster correction at normal density, due to Brueckner and Gammel (1947):

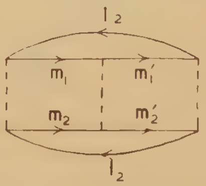
$$(\Delta E/A)_{\mathrm{pp}} \approx +0{\cdot}3\ \text{MeV}.$$

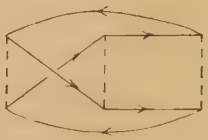
The context suggests that this has been obtained from a K matrix resulting from numerical solution of the appropriate equation with a realistic nuclear potential.

---

† Actually he quotes a somewhat smaller value due to the use of an effective mass (§ 3.7).

The hole–hole scattering diagram (3.79) has direct and exchange components





Together these give an energy shift

$$(\Delta E)_{\mathrm{hh}} = \tfrac{1}{2}\sum \frac{\langle \mathbf{m}_1\mathbf{m}_2|K|\mathbf{l}_1\mathbf{l}_2\rangle\langle \mathbf{l}_1\mathbf{l}_2|K|\mathbf{m}_1'\,\mathbf{m}_2'\rangle\langle\mathbf{m}_1'\mathbf{m}_2'|K|\mathbf{m}_1\mathbf{m}_2-\mathbf{m}_2\mathbf{m}_1\rangle}{(m_1'^2+m_2'^2-l_1^2-l_2^2)(m_1^2+m_2^2-l_1^2-l_2^2)/4M^2}\,. \qquad . \;\; . \;\; . \quad (3.87)$$

This could be treated much as (3.78). However, the only estimate in the literature proceeds in another way. Consider the expression

$$-\tfrac{1}{2}\sum \frac{\langle \mathbf{m}_1\mathbf{m}_2|\,K\!\left(\dfrac{m_1^2}{2M}+\dfrac{m_2^2}{2M}\right)|\,\mathbf{m}_1'\mathbf{m}_2'\rangle\langle\mathbf{m}_1'\mathbf{m}_2'|\,K\!\left(\dfrac{m_1^2}{2M}+\dfrac{m_2^2}{2M}\right)|\,\mathbf{m}_1\mathbf{m}_2-\mathbf{m}_2\mathbf{m}_1\rangle,}{(m_1^2+m_2^2-m_1'^2-m_2'^2)/2M} \qquad . \;\; . \;\; . \quad (3.88)$$

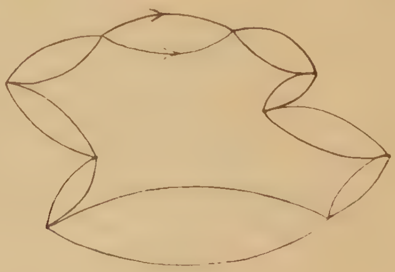
where the singularity is excluded in the summation. Although apparently of second order in K, this is actually of third order, for, if the $\mathscr{E}$ dependence of K is ignored the summation vanishes identically because of the antisymmetry between the **m**'s and **m**′'s. To second order one readily sees that

$$K(\mathscr{E})-K(\mathscr{E}')=K\left(\frac{Q}{\mathscr{E}-\mathscr{H}_0}-\frac{Q}{\mathscr{E}'-\mathscr{H}_0}\right)K \qquad . \;\; . \quad (3.89)$$

where $\mathscr{H}_0$ is the two-particle kinetic energy operator. Using this (3.88) is seen to be equivalent in third order to (3.87). Moszkowski and Sessler (1960) evaluate (3.88) for the K matrix resulting from the work of Moszkowski and Scott. They find an energy shift per particle

$$(\Delta E/A)_{\mathrm{hh}} \approx 0{\cdot}5\ \text{MeV}.$$

It is of some interest that the hole–hole scattering diagram can be included along with the particle–particle scattering in a sequence of diagrams known as 'generalized ladders'. These are of the form



where the interactions (other than that at zero time) may occur in any relative order). We discuss later (§ 4.3) the summation of all such diagrams due to Prange (1960). According to Iwamoto (1959, 1960) an approximate summation is effected by replacing the original K matrix by K'' determined from

$$K''(\mathscr{E}) = v + v\frac{Q-P}{\mathscr{E}-\mathscr{H}_0}K''(\mathscr{E}). \qquad (3.90)$$

Here P projects *below* the fermi surface. Equivalently

$$\begin{aligned} K''(\mathscr{E}) &= K(\mathscr{E}) + K(\mathscr{E})\frac{-P}{\mathscr{E}-\mathscr{H}_0}K''(\mathscr{E}) \\ &= K(\mathscr{E}) - K(\mathscr{E})\frac{P}{\mathscr{E}-\mathscr{H}_0}K(\mathscr{E}) + K(\mathscr{E})\frac{P}{\mathscr{E}-\mathscr{H}_0}K(\mathscr{E})\frac{P}{\mathscr{E}-\mathscr{H}_0}K(\mathscr{E}) \\ &\quad + \dots . \end{aligned}$$

The term *apparently* of second order here is just the third-order term of the preceding paragraph. Indeed Moszkowski and Sessler started from Iwamato's equation. Fortunately they did not proceed to the *explicitly* third-order term, which is fictitious†.

It remains to consider the insertion of self-energy parts into second-order diagrams, as in (3.76). The insertion may be into any of the four original lines. Constructing the corresponding energy shift in perturbation theory, but with matrix elements of K rather than v, yields

$$\begin{aligned} (\Delta E)_{\text{s.e.}} &= \tfrac{1}{2}\sum \frac{\langle \mathbf{m}_1\mathbf{m}_2|K|\mathbf{l}_1\mathbf{l}_2\rangle\langle \mathbf{l}_1\mathbf{l}_2|K|\mathbf{m}_1\mathbf{m}_2\rangle}{(m_1{}^2+m_2{}^2-l_1{}^2-l_2{}^2)^2/4M^2}\{V(l_1)+V(l_2)-V(m_1)-V(m_2)\} \end{aligned} \qquad (3.91)$$

where

$$V(k) = \sum_{\mathbf{m}} \langle \mathbf{km}|K|\mathbf{km}-\mathbf{mk}\rangle \qquad (3.92)$$

The quantity V is a generalization of the effective one-body potential of Hartree–Fock theory, which is obtained with v rather than K in (3.92). To the order of accuracy considered the term (3.91) may be incorporated in the original formula (3.41) by the substitution

$$\left.\begin{aligned} m^2/2M &\to m^2/2M + V(m), \\ l^2/2M &\to l^2/2M + V(l), \end{aligned}\right\} \qquad (3.93)$$

in (3.39) and (3.41). This means in effect that a particle propagates always in a mean field, due to other particles, which modifies its energy. This is what we have meant by occasional references to 'self-consistent field effects'. If the matrix elements of K are replaced by those of a regular central potential the energy shift (3.91) is quite small. According to Martin and de Dominicis (1957) it is no more important than other third-order effects. However, with more realistic potentials the result is rather larger. Moszkowski and Scott (1960), with a central Serber potential with hard core, find a shift of about 5 Mev per particle at normal density. With a still more realistic potential (that of Brueckner and Gammel 1958) their estimate greatly increases (private communication). It is then in substantial agreement with that obtained by Brueckner and Gammel, as quoted at the end of § 3.5. The increase is attributed mainly to the inclusion of the tensor force.

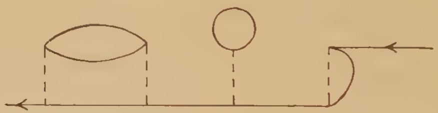
---

† Like Moszkowski and Sessler we assume here that singular integrals are to be defined with principal values. Iwamoto's intention is not clear and his derivation suggests that the point is one of some delicacy.

The general situation then is this. In so far as corrections have been estimated, only the self-consistent field effect appears to be substantial. This certainly cannot be omitted and may be too large to be adequately treated as a small correction. The latter view is taken by Brueckner and Gammel and motivates their approach in which self-consistent field effects are built in at the start. To this quite elaborate formalism we now turn.

3.7. *The Self-consistent Field*

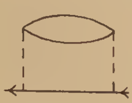
The systematic introduction of the self-consistent field depends, in the language of field theory, on the elimination of self-energy parts. We work here entirely with Feynman (as distinct from time-ordered) diagrams. A self-energy part in one such as

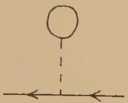


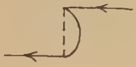
. . . (3.94)

which becomes disconnected from the rest of the diagram when two solid lines are broken. It can be regarded as the result of inserting additional pieces in a single line. The 'skeleton' of any diagram is obtained by removing every portion that can be regarded as such a self-energy insertion, with the proviso that the special interaction at $t = 0$ must not be so removed. The family of diagrams with a given skeleton is obtained by making all possible insertions independently into all solid lines of the skeleton. By associating with such a line not the original quantity $G^0(x, x')$, but a new quantity $G(x, x')$, obtained by summing over all possible insertions, the contribution of the whole family of diagrams is given at once. To indicate that G rather than G^0 is implied, we draw skeleton diagrams with heavy lines thus

Self-energy parts are called 'proper' if they cannot be divided into two or more parts connected only by a single line. The diagram (3.94) is improper, being a succession of three proper parts







The contribution to $G(x, x')$ from proper insertions only we write in the form

$$\int dx''\, dx'''\, G^0(x, x'')\, \langle x'' \,|\, -iM \,|\, x''' \rangle\, G^0(x''', x'). \tag{3.95}$$

Here the factors G^0 arise from the terminal lines and $(-iM)$ is a sum over all possible proper insertions. It is usually convenient to regard the variables x, x', etc., as generalized matrix indices and to suppress them as in matrix algebra. Then (3.95) is written simply

$$G^0(-iM)\,G^0. \tag{3.96}$$

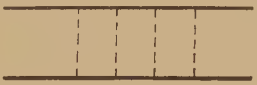
With this notation the complete propagator, the result of making any number of proper insertions, is clearly

$$G = G^0 - G^0\, iMG^0 + G^0\, iMG^0\, iMG^0 - \ldots\,.$$

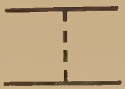
This is the iterative solution of either of the integral equations

$$\left.\begin{aligned} G &= G^0 - i\,G^0\,MG, \\ G &= G^0 - i\,G\,M\,G^0. \end{aligned}\right\} \tag{3.97}$$

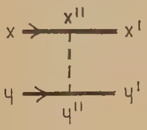
It is useful also to eliminate 'ladder' parts, i.e. those of the type



The sum of such ladders with any number of interaction lines we denote by a single heavy interaction line,



To obtain the appropriate factor note that the basic diagram



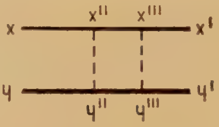
makes a contribution which can be written

$$\int dx''\,dy''\,dx'''\,dy'''\,G(x,x'')\,G(y,y'')\langle x''y''\,|-iv\,|x'''y'''\rangle G(x''',x')\,G(y''',y') \qquad . \quad . \quad . \quad (3.98)$$

where

$$\langle xy\,|v\,|x'y'\rangle = v(\mathbf{x}-\mathbf{y})\,\delta(x_0-y_0)\,\delta(x_0'-y_0')\,\delta(x_0-x_0')\,\delta(\mathbf{x}-\mathbf{x}')\,\delta(\mathbf{y}-\mathbf{y}') \qquad . \quad . \quad . \quad (3.99)$$

where x_0 is the time component of x, etc. The next contribution



can be obtained by making in (3.98) the replacement

$$\langle xy\,|-iv\,|x'y'\rangle \to \int dx''\,dy''\,dx'''\,dy'''$$
$$\times\,\langle xy\,|-iv\,|x''y''\rangle\,G(x'',x''')\,G(y'',y''')\langle x'''y'''\,|-iv\,|x'y'\rangle$$

or in an obvious matrix notation

$$-iv \to (-iv)GG(-iv).$$

In this notation the sum of arbitrarily long ladders is obtained with the replacement

$$-iv \to -iK'$$
$$K' = v + v(-iGG)v + v(-iGG)\,v(-iGG)\,v$$
$$+\ldots.$$

The matrix K' is thus the solution of either of the integral equations

$$\left.\begin{aligned} K' &= v - ivGG\,K', \\ K' &= v - iK'\,GG\,v. \end{aligned}\right\} \qquad . \quad . \quad . \quad . \quad . \quad . \quad (3.100)$$

The quantity K' is related to the Brueckner reaction matrix, but the relation is complicated by the fact that the latter was defined for ladders in time-ordered rather than Feynman diagrams. To spell out the connection note first that in all orders $\langle xy\,|K'\,|x'y'\rangle$ contains the factor $\delta(x_0-y_0)\,\delta(x_0'-y_0')$; so there is really only one initial and one final time argument. Define retarded and advanced parts of the propagator G by

$$\left.\begin{aligned} G^+(x,x') &= \theta(x_0-x_0')\,G(x,x'), \\ G^-(x,x') &= \theta(x_0'-x_0)\,G(x,x'), \end{aligned}\right\} \qquad . \quad . \quad . \quad . \quad (3.101)$$

where

$$\theta(t) = \tfrac{1}{2} - \tfrac{1}{2}\,t/|t|.$$

Define a new quantity K by

$$\begin{aligned} K &= v - iv\,G^+G^+K \\ &= v - iK\,G^+G^+v. \qquad \ldots\ldots\ldots \quad (3.102) \end{aligned}$$

This will have a momentum representation of the form

$$\langle xy\,|\,K\,|\,x'y'\rangle = \delta(x_0 - y_0)\,\delta(x_0' - y_0')$$
$$\sum \langle \mathbf{k}_1\mathbf{k}_2\,|\,K(x_0, x_0')\,|\,\mathbf{k}_1'\,\mathbf{k}_2'\rangle \exp(-i\mathbf{k}_1\,.\,\mathbf{x} - i\mathbf{k}_2\,.\,\mathbf{y} + i\mathbf{k}_1'\,.\,\mathbf{x}' + i\mathbf{k}_2'\,.\,\mathbf{y}').$$

Finally introduce the transformed operator $K(\mathscr{E})$ by

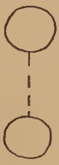
$$K(\mathscr{E}) = \int_{-\infty}^{0} dt \exp(-i\mathscr{E}t)\,K(0, t). \qquad \ldots\ldots \quad (3.103)$$

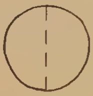
Then it is readily seen that in the approximation $G = G^0$, $K(\mathscr{E})$ is identical with the operator previously so labelled (3.39). Note from (3.100), (3.102) that

$$\begin{aligned} K' &= K - iKG^-G^-K' \\ &= K - iK'G^-G^-K. \qquad \ldots\ldots\ldots \quad (3.104) \end{aligned}$$

To first order in K, $K' = K$. This approximation is much used in the following.

In analogy with (3.35) the first diagrams of the new theory are





and the corresponding energy shift is

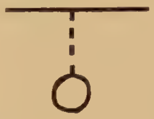
$$\Delta E = \tfrac{1}{2}\int d\mathbf{x}\,d\mathbf{y}\,d\mathbf{x}'\,d\mathbf{y}'\,\langle xy\,|\,K'\,|\,x'y'\rangle\{G(x', x)\,G(y', y) - G(x', y)\,G(y', x)\} \qquad \ldots \quad (3.105)$$

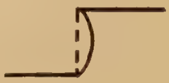
with $x_0 = y_0 = 0$ and integration over only negative values of x_0', y_0'. The corrections to this will not include the large self-consistent field terms, already incorporated. The other third-order corrections will have somewhat modified values, but should remain small.

The eqn. (3.105) does not become useful until some practical approximations for K' and G are defined. Here we derive the equations used in the work of Brueckner and Gammel (1958). This derivation is much indebted to the work of Klein and Prange (1958) and Thouless (1959 a, b).

The principal approximation is to calculate M only to first order in K. Then G is computed from M by (3.97), and K from G by (3.102). It is also expedient to make the approximation that G, like G^0, is translationally invariant in time—i.e. a function only of the difference to the two time arguments. This property does not hold accurately because the time origin occurs explicitly in the formalism, as the upper limit of all time integrations. This assumed time-translation symmetry of G is communicated automatically to K and M.

In this scheme M is obtained from the lowest order diagrams





with the approximation $K'=K$. The result is

$$\langle x\,|\,M\,|\,x'\rangle=\int dy\,dy'\,G(y,y')\{\langle xy\,|\,K\,|\,x'y'\rangle-\langle xy\,|\,K\,|\,y'x'\rangle\}. \qquad (3.106)$$

The momentum representation of this and similar quantities (diagonal by space-translation symmetry) is of the form

$$\langle x\,|\,M\,|\,x'\rangle=\sum_{\mathbf{k}}\exp\,[i\mathbf{k}\,.\,(\mathbf{x}-\mathbf{x}')]\,M_k\,(x_0,x_0').$$

Like K, M is a retarded operator, i.e. zero for $x_0<x_0'$. Because of the time translation symmetry (in the present approximation scheme)

$$M_k(t,t')=M_k\,(0,t'-t),$$

so M_k is entirely determined by the transform

$$M_k(\mathscr{E})=\int_{-\infty}^{0}dt\,M_k\,(0,t)\exp\,(-i\mathscr{E}t). \qquad (3.107)$$

In momentum space (3.97) becomes

$$G_k(t,t')=G_k{}^0(t,t')-i\int dt''\,dt'''\,G_k{}^0(t,t'')\,M_k(t'',t''')\,G_k(t''',t'). \qquad (3.108)$$

For momenta below the fermi surface, and for $t'=0$, it can be confirmed by substitution that the solution is

$$G_m(t,0)=-\theta(-t)\exp\,(-ie_mt) \qquad (3.109)$$

where

$$e_m=\frac{m^2}{2M}+M_m\,(e_m). \qquad (3.110)$$

For $t'<0$ we use the translated expression

$$G_m(t,t')=-\theta(t'-t)\exp\,[-ie_m(t-t')]. \qquad (3.109\,a)$$

Now this is not a solution of (3.108). It solves an equation (Klein and Prange 1958, Thouless 1959 a, b) in which the restriction $t''<0$ is replaced by $t''<t'$. This mutilation can be defended on two grounds: (*a*) It would in effect occur automatically if $K(t,t')$ and so $M(t,t')$ contained a factor $\delta(t-t')$, or equivalently if $K(\mathscr{E})$ were independent of $\mathscr{E}$. The error in

pretending that this is so is of second order in K. (*b*) Because one interaction occurs always at zero time, the time argument $t=0$ is of particular importance. For this reason the error in (3.105) due to the use of (3.110) is actually of fifth order in K.

For momenta above the fermi surface G_l^0 is, like M_l, a retarded quantity. As a result the time integrations in (3.108) do not extend up to the origin, and G_l has time translation symmetry without further approximation. It is also a retarded operator, and so fully determined by the transform

$$G_l(\mathscr{E})=\int_{-\infty}^{0} dt \exp(i\mathscr{E}t)\, G_l(0,t). \qquad (3.111)$$

From (3.108)

$$\begin{aligned} G_l(\mathscr{E}) &= G_l^0(\mathscr{E})\{1+iG_l^0(\mathscr{E})\, M_l(\mathscr{E})\}^{-1} \\ &= i\{\mathscr{E}-l^2/2M-M_l(\mathscr{E})\}^{-1}. \qquad (3.112) \end{aligned}$$

The result (3.109) has a simple interpretation. For propagation below the fermi surface, in this approximation, the effect of the medium is to add to the kinetic energy a momentum dependent potential with matrix elements $M_m(e_m)$. The dependence of M_l on $\mathscr{E}$ in (3.112) prevents the inversion of the transform from giving a simple exponential for G_l. As usual this $\mathscr{E}$ dependence is of second order in K, so that $G_l(t,t')$ is approximately exponential.

Next we calculate K from (3.102). Taking the transform we obtain

$$K(\mathscr{E})=v-iv\,L(\mathscr{E})\,K(\mathscr{E}), \qquad (3.113)$$

where

$$\langle ll' \,|\, L(\mathscr{E}) \,|\, ll' \rangle=\int_{-\infty}^{0} dt \exp(-i\mathscr{E}t)\, G_l(0,t)\, G_{l'}(0,t), \qquad (3.114)$$

and L has no matrix elements involving momenta below the fermi surface. This integral can of course be expressed as a convolution of the transforms (3.112) of the G's. However, if we approximate G_l by an exponential the integral is trivial. Suppose that e_l can be chosen so that

$$\Delta_l(t)=G_l(0,t)-\exp(ie_l t) \qquad (3.115)$$

is small. Ignoring terms quadratic in Δ,

$$\langle ll' \,|\, L(\mathscr{E}) \,|\, ll' \rangle=\int_{-\infty}^{0} dt \exp(-i\mathscr{E}t)\{\exp[i(e_l+e_{l'})t]+\Delta_l(t)\exp(ie_{l'}t) + \Delta_{l'}(t)\exp(ie_l t)\}.$$

The terms linear in Δ vanish if

$$\begin{aligned} 0 &= \{\mathscr{E}-e_{l'}-l^2/2M-M_l(\mathscr{E}-e_{l'})\}^{-1}-\{\mathscr{E}-e_l-e_{l'}\}^{-1} \\ &\quad +\{\mathscr{E}-e_l-l'^2/2M-M_{l'}(\mathscr{E}-e_l)\}^{-1}-\{\mathscr{E}-e_{l'}-e_l\}^{-1}. \end{aligned}$$

Allowing e_l to depend on $\mathscr{E}$ and l' we have the solution

$$e_l(l',\mathscr{E})=l^2/2M+M_l(\mathscr{E}-e_{l'}(l,\mathscr{E})), \qquad (3{\cdot}116)$$

and likewise with l, l' interchanged. That e_l should depend on the additional arguments indicates simply that the best exponential fit depends on where the propagator is to be used. The final equation for K is

$$K(\mathscr{E}) = v + v \sum_{\mathbf{ll}'} \frac{|\mathbf{ll}'\rangle\langle\mathbf{ll}'|}{\mathscr{E} - e_l(l', \mathscr{E}) - e_{l'}(l, \mathscr{E})} K(\mathscr{E}). \quad . \quad . \quad (3.117)$$

Using (3.109) to rewrite (3.106) as,

$$M_k(\mathscr{E}) = \sum_{\mathbf{m}} \langle \mathbf{mk} \,|\, K(\mathscr{E} + e_m) | \,\mathbf{mk} - \mathbf{km} \rangle, \quad . \quad . \quad (3.118)$$

the circle of eqns (3.117), (3.116), (3.118) and (3.110) is complete. These equations are equivalent to those in Appendix A of Brueckner and Gammel (1958). They appear to have arrived at them (actually at the less compact version in their main text) in a largely intuitive way.

A simpler set of equations is obtained by using once more the slow variation of $M(\mathscr{E})$ with $(\mathscr{E})$. Equation (3.116), differs from (3.110) only in the argument of M. Apart from an error of second order we can take for all k

$$e_k = k^2/2M + M_k(e_k) = k^2/2M + V_k. \quad . \quad . \quad . \quad (3.119)$$

Then e_l depends on l only, not on l' and $\mathscr{E}$. Only the argument $\mathscr{E} = e_k$ is now needed in $M_k(\mathscr{E})$. and the matrix elements of K required in (3.118) are on the energy shell. For $k < k_f$ the K matrix equation now involves a vanishing energy denominator, and the principal part of the singular integral is specified. The quantity V_k is a generalization of the Hartree–Fock potential. We call this approximation that of the 'effective field'. It differs from the previous approximation by a term in the energy which is fourth order.

A further simplification once popular is to approximate V_k by a quadratic form

$$V_k = a + bk^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.120)$$

Then

$$e_k = a + bk^2 + k^2/2M$$

$$= a + k^2/2M^* \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.121)$$

where M^* is the 'effective mass' (Weisskopf 1957). In the K matrix equation only differences of e_k's occur, and the only effect of the modified propagation is to substitute M^* for the original mass. For nuclear matter this effective mass approximation is poor (Brueckner 1955 a, Martin and de Dominicis 1957 a). Typically V_k turns out to have the form shown in fig. 2.

Important transitions go from below k_f to well above, where V_k is negligible. The effective mass approximation, chosen for example to give the slope correctly at k_f, greatly overestimates the increase due to V of the corresponding energy denominators.

Finally, from (3.105) and (3.110), the energy shift is

$$\Delta E = \tfrac{1}{2} \sum_{\mathbf{m}\,\mathbf{m}'} \langle \mathbf{m}\mathbf{m}' \,|\, K(e_m + e_{m'}) \,|\, \mathbf{m}\mathbf{m}' - \mathbf{m}'\mathbf{m} \rangle \quad . \quad (3.122)$$

$$= \tfrac{1}{2} \sum_{\mathbf{m}} M_m(e_m) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3.123)$$

$$= \tfrac{1}{2} \sum_{\mathbf{m}} [e_m - m^2/2M]. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3.124)$$

The last equation is as in the Hartree–Fock theory; the total energy shift is one half the sum of 'level shifts' of individual occupied states.

3.8. *The Brueckner–Gammel Calculation*

Attempts to apply the self-consistent theory have been made mainly by Brueckner and his collaborators. Here we pass over early attempts using simplified forces and the phase shift approximation to the K matrix (Brueckner and Wada 1956, and references therein). We come at once to the work of Brueckner and Gammel (1958), the most ambitious attempt on the nuclear matter problem yet published†. This uses a quite realistic potential in conjunction with a simplied version of the theory contained in equations (3.117), (3.116), (3.118), (3.110) and (3.124). To recapitulate, the energy shift is given by

$$\Delta E = \tfrac{1}{2} \sum_{\mathbf{m}} (e_m - m^2/2M) \quad . \; . \; . \; . \; . \; . \quad (3.125)$$

where

$$e_m = m^2/2M + \sum_{\mathbf{m}'} \langle \mathbf{m}\mathbf{m}' \,|\, K(e_m + e_{m'}) \,|\, \mathbf{m}\mathbf{m}' - \mathbf{m}'\mathbf{m} \rangle, \quad . \quad (3.126)$$

$$K(\mathscr{E}) = v + \sum_{\mathbf{l}\,\mathbf{l}'} \frac{v \,|\, \mathbf{l}\mathbf{l}' \rangle \langle \mathbf{l}\mathbf{l}' \,|\, K(\mathscr{E})}{\mathscr{E} - e_l(l', \mathscr{E}) - e_{l'}(l, \mathscr{E})} \quad . \; . \; . \quad (3.127)$$

$$e_l(l', \mathscr{E}) = l^2/2M + \sum_{\mathbf{m}} \langle \mathbf{l}\mathbf{m} \,|\, K(\mathscr{E} + e_m - e_{l'}(l, \mathscr{E})) \,|\, \mathbf{l}\mathbf{m} - \mathbf{m}\mathbf{l} \rangle. \quad (3.128)$$

To simplify these equations Brueckner and Gammel replaced $\{e_m - e_{l'}(l, \mathscr{E})\}$ in (3.128) by some 'suitable' average value $(-\Delta)$. The quantity $e_l(l', \mathscr{E})$ defined by (3.128) then becomes independent of l'. We use the new notation

$$e_l(l', \mathscr{E}) \to \bar{e}(\mathscr{E}, l)$$

so that

$$\bar{e}(\mathscr{E}, l) = l^2/2M + \sum_{\mathbf{m}} \langle \mathbf{l}\mathbf{m} \,|\, K(\mathscr{E} - \Delta) \,|\, \mathbf{l}\mathbf{m} - \mathbf{m}\mathbf{l} \rangle. \quad . \; . \; . \quad (3.129)$$

The matrix $K(\mathscr{E})$ is then required for $\mathscr{E}$ equal to all values of $(e_m + e_{m'} - \Delta)$, i.e. from $(2e_0 - \Delta)$ to $(2e_f - \Delta)$ where e_0 and e_f are the values of e_m with $m = 0$ and $m = k_f$ respectively. Variation of Δ from zero to $(e_f - e_0)$—about 70 MeV—increases the equilibrium spacing by about 4% and the energy per particle by about 1 MeV. Brueckner and Gammel believe $\Delta = (e_f - e_0)$ is about right and quote final results with this value. Lacking their insight, we cannot see why twice this value (for example) should not be equally reasonable, and we would attribute a corresponding uncertainty to the result. It should be said that this approximation, i.e. replacing (3.128) by (3.129), does not seem so essential now as when the work was begun. At that time they had only an elaborate series of equations involving an infinite hierarchy of different K matrices. It was later observed (Brueckner and Gammel 1958, Appendix A) that the whole hierarchy could be expressed in terms of the one matrix $K(\mathscr{E})$ involving the extra parameter $\mathscr{E}$.

---

† However, calculations have been announced by Martin and Puff (1960) and by Moszkowski and Scott (private communication) which use the same potential and get similar results, apparently with less labour.

The numerical solution of the equations is by the obvious iterative process. At first the energies are given the unperturbed values

$$e_m = m^2/2M, \quad \bar{e}(\mathscr{E}, l) = l^2/2M.$$

The K matrix equation is then that already considered in § 3.5, and is solved numerically as described at the end of that section. New approximations to e_m and $\bar{e}(\mathscr{E}, l)$ are then obtained from (3.126) and (3.129), and a first approximation (with no self-consistency effects) to the energy shift is obtained from (3.125). The process is restarted with the new values of e_m and $\bar{e}(\mathscr{E}, l)$. On this second time round the modified energy denominators would be a new cause, in addition to the exclusion principle, for coupling between partial waves. Again this is avoided by spherical averaging. For intermediate states of momentum $(\frac{1}{2}\mathbf{p} \pm \mathbf{k})$ the propagator is

$$f(\mathbf{p}, \mathbf{k})\{\mathscr{E} - \bar{e}[\mathscr{E}, \sqrt{(\tfrac{1}{4}p^2 + k^2 + \mathbf{k}\,.\,\mathbf{p})}] - \bar{e}[\mathscr{E}, \sqrt{(\tfrac{1}{4}p^2 + k^2 - \mathbf{k}\,.\,\mathbf{p})}]\}^{-1}. \quad (3.130)$$

For small momenta

$$\bar{e}\left(\mathscr{E}, \left|\frac{\mathbf{p}}{2} \pm \mathbf{k}\right|\right)$$

is linear in $(\frac{1}{2}\mathbf{p} \pm \mathbf{k})^2$ and the $\mathbf{k}\,.\,\mathbf{p}$ terms in (3.130) cancel. This is again so at high momenta, where $\bar{e}(\mathscr{E}, k)$ is essentially just the kinetic energy $k^2/2M$. To make some allowance for $\mathbf{k}\,.\,\mathbf{p}$ in the intermediate region it is treated as a perturbation. To second order in $\mathbf{k}\,.\,\mathbf{p}$ we then need the average over $(\mathbf{k}\,.\,\mathbf{p}/kp)$ of

$$\frac{f(\mathbf{p}, \mathbf{k})}{\mathscr{E} - 2\bar{e}[\mathscr{E}, \sqrt{(k^2 + \frac{1}{4}p^2)}]} + \frac{f(\mathbf{p}, \mathbf{k})(\mathbf{k}\,.\,\mathbf{p})^2 \ddot{\bar{e}}[\mathscr{E}, \sqrt{(k^2 + \frac{1}{4}p^2)}]}{[\mathscr{E} - 2\bar{e}[\mathscr{E}, \sqrt{(k^2 + \frac{1}{4}p^2)}]]^2},$$

where $\ddot{\bar{e}}(\mathscr{E}, l)$ is the second derivative of $\bar{e}(\mathscr{E}, l)$ with respect to l. With $\bar{f}$ from (3.67), and

$$\overline{f(\mathbf{p}\,.\,\mathbf{k})\,(\mathbf{k}\,.\,\mathbf{p})^2} = \tfrac{1}{3}(kp)^2 \bar{f}^2, \quad . \quad . \quad . \quad . \quad . \quad (3.131)$$

and reabsorbing the correction into the denominator, we have the averaged Green function

$$G(\mathbf{r}, \mathbf{r}')$$
$$= \sum_{\mathbf{k}} \frac{\bar{f}(k, p) \exp(i\mathbf{k}\,.\,(\mathbf{r} - \mathbf{r}')}{\mathscr{E} - \bar{e}\left[\mathscr{E}, \sqrt{\left(k^2 + \frac{1}{4}p^2 + \frac{1}{\sqrt{3}}kp\bar{f}\right)}\right] - \bar{e}\left[\mathscr{E}, \sqrt{\left(k^2 + \frac{1}{4}p^2 - \frac{1}{\sqrt{3}}kp\bar{f}\right)}\right]}.$$

This is the form (resolved into partial waves) used by Brueckner and Gammel, except that instead of $\bar{f}(k,p)$ in the denominator they have $\bar{f}^2$; Brueckner (1958) has $\bar{f}^{3/2}$. When the calculation was repeated at Cambridge (Clement *et al.* 1959) it was found that the distinction between $\bar{f}^2$ and $\bar{f}^{3/2}$ was negligible. Presumably then $\bar{f}$ gives nearly the same results, and indeed perhaps this term can be omitted.

About five iterations were sufficient to obtain constant results. Very coarse meshes were used (for example e_m was considered at only five values of m between 0 and k_f). This could be reduced to about forty-five minutes by using still cruder meshes and certain integration formulae. The major item in this numerical work was the construction of the partial wave Green functions.

Their results are quoted in table 3 for two potentials B and C. Both fit low energy data, have hard cores, spin dependent Yukawa central forces, and tensor forces; C has also a spin-orbit coupling, which improves the fit to high energy scattering data. In the preferred case, C, the binding energy and equilibrium spacing are closely equal to the desired values. In view of the many approximations in the theory these results might be thought adequate. Brueckner and Gammel observe that a crude allowance for coulomb repulsion in real heavy nuclei improves the agreement. As observed in § 1.1 an equally crude allowance for surface tension gives a larger effect in the opposite direction.

Table 3. Properties of nuclear matter as calculated by Brueckner and Gammel

	Potential B	Potential C
Binding energy (MeV)	18·5	15·2
Equilibrium spacing r_0 (f)	0·95	1·02
Compressibility (MeV)	167	172
M^* at fermi surface	0·72M	0·73M
Symmetry energy (MeV)	32	22

Theoretical values are given by Brueckner and Gammel also for compressibility, symmetry energy, and 'effective mass at the fermi surface'. The latter, defined by

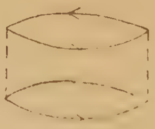
$$(M^*)^{-1} = 2(d/dm^2)\, e_m \big|_{m^2=k_f^2} \quad \ldots \quad (3.133)$$

has no direct physical significance (§ 3.9). There is no clear empirical result with which to compare the predicted compressibilities (§ 1.7). The empirical symmetry energy is intermediate between the two quoted values. The difference of these is attributed mainly to the inclusion in C of odd state central interaction. The calculation of symmetry energy allows only for the fact that the summation in (3.122) is below different fermi surfaces for neutrons and protons. It ignores the dependence of the K matrix itself on the neutron–proton ratio. A similar approximation in the discussion of separation energy (§ 3.9) gives a quite wrong result.

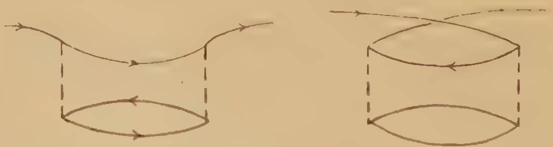
3.9. *Separation and Rearrangement Energies*

In the Hartree–Fock approximation there is a celebrated theorem which states that the fictitious single particle energy of the topmost occupied level has a real physical significance in a large system. It is then the least energy required to remove a particle from the system—the 'separation energy'. The question arises as to whether the quantity e_f of Brueckner theory has a similar significance. The answer is no, as we will see. We come later (§ 3.10) to the generalization of the above theorem by Hugenholtz and Van Hove, but this does not refer to e_f. The difference between e_f and the separation energy, say e_f', been called 'rearrangement energy'.

In developing the perturbation series for the energy of the A particle system we assumed closed shells, i.e. all states occupied up to the fermi surface. Suppose now that one of the topmost states is unoccupied, and apply the same theory. The result is tentatively identified as the ground state of the perturbed $(A-1)$ particle system. The unperturbed state is now degenerate with many others in general (with different directions for the hole momentum or different spins), but because of conservation laws this is no complication†. The series for the energy of the $(A-1)$ system differs from that of the A system merely in that **l** (particle) summations extend over an extra state and **m** (hole) summations over one less. The difference, the separation energy, can be represented by breaking one or more lines of the original diagram to indicate that they refer to the special state and must not be summed over. In a large system it is in general necessary to break only one line of a diagram at a time, for each such breakage removes a factor Ω associated with a summation. Thus the original diagram



gives rise to the separation energy diagrams

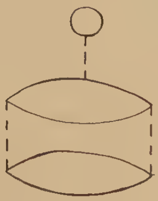


. . (3.134)

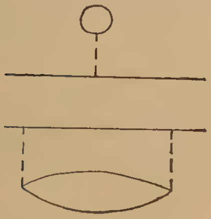
---

† We have of course momentum and isobaric spin conservation; parity conservation is sufficient to show that states of different ordinary spin do not mix.

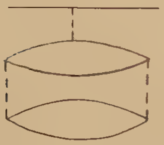
However if two lines refer necessarily to the same state they must be broken simultaneously. Thus



gives rise, among others, to



(3.135)



(3.136)

We will later (§ 3.10) see a construction of the separation energy that does not involve these disconnected diagrams, but which does not give an explicit series.

The series just described can be constructed essentially by differentiating the original energy series with respect to k_f. The only difference is that spins and momentum directions are then automatically averaged over, but this should not matter in a large system. Thus if $E(k_f)$ is the sum of the original energy series

$$\begin{aligned} e_f' &= -\frac{\partial E(k_f)}{\partial k_f}\frac{\partial k_f}{\partial A} \\ &= -\left(\frac{1}{4\pi}\right)\frac{(2\pi)^3}{4\pi\, k_f^2}\frac{1}{\Omega}\left\{\frac{\partial E(k_f)}{\partial k_f}\right\}. \qquad (3.137) \end{aligned}$$

We could in the same way have defined a quantity e_f'' as the energy gain on *adding* a particle to the lowest unfilled shell. By similar reasoning we would have arrived again at the value (3.137), so that in a large system $e_f' = e_f''$. Note that this result has been established only for perturbation expansions and may fail when these are inapplicable as in the theory of superconductivity (§ 4.4).

When the system is at equilibrium density the energy per particle is stationary as a function of k_f, i.e.

$$\frac{\partial}{\partial k_f}\left(\frac{E}{A}\right) = \frac{1}{A}\left(\frac{\partial E}{\partial k_f} - \frac{E}{A}\frac{\partial A}{\partial k_f}\right) = 0.$$

Then (3.137) is equivalent to

$$e_f' = E/A, \qquad (3.138)$$

i.e. the separation energy is the same as the energy per particle. In the work of Brueckner and Gammel, at equilibrium density, $e_f = -28$ MeV and $E/A = -15$ MeV so the rearrangement energy is 12 MeV.

In the Brueckner approximation

$$E = \sum_{\mathbf{m}} \frac{m^2}{2M} + \tfrac{1}{2} \sum_{\mathbf{m}\,\mathbf{m}'} \langle \mathbf{m}\mathbf{m}' | K(e_{\mathbf{m}} + e_{\mathbf{m}'}) | \mathbf{m}\mathbf{m}' - \mathbf{m}'\mathbf{m} \rangle$$

so, from (3.137)

$$\begin{aligned} e_f' &= \frac{k_f^2}{2M} + \sum_{\mathbf{m}} \langle \mathbf{k}_f \mathbf{m} | K(e_f + e_m) | \mathbf{k}_f \mathbf{m} - \mathbf{m}\mathbf{k}_f \rangle \\ &\qquad + \tfrac{1}{2} \sum_{\mathbf{m}\,\mathbf{m}'} \langle \mathbf{m}\mathbf{m}' | \frac{\partial k_f}{\partial A} \frac{\partial}{\partial k_f} K(e_m + e_{m'}) | \mathbf{m}\mathbf{m}' - \mathbf{m}'\mathbf{m} \rangle \\ &= e_f + \tfrac{1}{2} \sum_{\mathbf{m}\,\mathbf{m}'} \langle \mathbf{m}\mathbf{m}' | \frac{\partial k_f}{\partial A} \frac{\partial}{\partial k_f} K(e_m + e_{m'}) | \mathbf{m}\mathbf{m}' - \mathbf{m}'\mathbf{m} \rangle. \qquad (3.139) \end{aligned}$$

The last term in (3.139), which is the rearrangement energy, would be zero in the Hartree–Fock approximation where K reduces to v. More generally if we write the K-matrix equation in the form

$$K = v + v\,(Q/e)\,K$$

we see that an explicit dependence on k_f arises both from the exclusion principle operator Q and from the energies e_m and e_l in the denominator e. Retaining each of these effects to lowest order in K gives, for the change in K due to a change in k_f,

$$\delta K = K \frac{\delta Q}{e} K + KQ\delta\left(\frac{1}{e}\right) K. \quad . \quad . \quad . \quad . \quad (3.140)$$

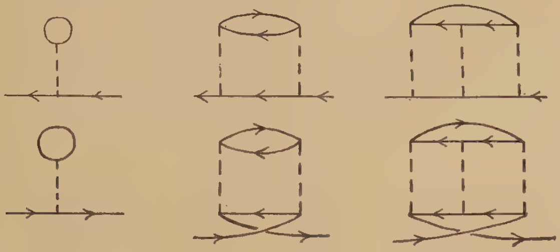
Using this in (3.139) we obtain second and third-order contributions analogous to diagrams (3.134) and (3.136), but with self-consistent propagators (§ 3.7) so that (3.135) does not arise. These correction terms were estimated by Brueckner *et al.* (1960) using the K matrix of Brueckner and Gammel (1958). They found

$$e_f' - e_f = 10 \text{ MeV}$$

in reasonable agreement with $E/A - e_f$ as we expect. It should be noted that any discrepancy here has nothing to do with the basic approximation of Brueckner theory, within which (3.138) and (3.139) are identities at equilibrium; it reflects only further approximations made in evaluating the two formulae (numerical, spherical averaging, the expansion (3.140), etc.). Various other estimates have been made of the terms arising from (3.140) (Thouless 1958, Squires 1958, Mittelstaedt 1958), but these have used crude approximations, to the K matrix for example, and are not quantitatively meaningful.

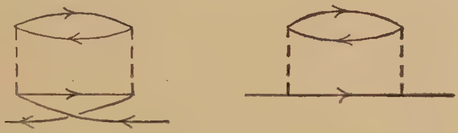
For the separation energy we have seen that the term of first order in K is subject to substantial correction. This raises the question as to whether the first-order treatment of the operator M in Brueckner theory is adequate.

This allows for such insertions in particle and hole lines (of time-ordered perturbation diagrams) as



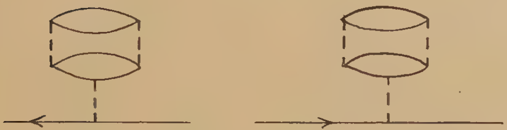
etc.

It does not allow for



etc., (3.141)

nor



etc. (3.142)

These last diagrams are clearly similar to the rearrangement energy diagrams discussed above. But they refer now to the propagation of virtual particles and holes and not to the energies of real particles and holes. The 'energies' of the virtual states are not well defined, for the propagators are not simple exponential functions. We saw in §§ 3.7 and 3.8 that only after a series of approximations could such energies be assigned. By further approximations of a similar type new virtual 'energies' can be defined which allow roughly for the additional insertions (Brueckner and Goldman 1960). Brueckner *et al.* (1960) find that the original energies are displaced by only a few Mev (5 at the fermi surface, rather than the 12 Mev for the removal of a real particle). This is substantially smaller than the first-order self-energy effect ($e_f - k_f^2/2M \approx -70$ Mev), which when introduced into K-matrix energy denominators increased the real mean energy per particle in nuclear matter by 20 Mev. Brueckner *et al.* find that the additional 'rearrangement' effect is only about 1·5 Mev.

3.10. *The Optical Model Potential*

Here again we consider the addition (or removal) of a particle to (or from) the closed shell system. Now we remove the restriction that the particle is added just above the fermi surface, or removed from just below. The general case presents the difficulty that the unperturbed state is degenerate, or nearly so, with many states in which one or more other particles are excited at the expense of the energy of the extra particle. The precise pattern of degeneracy shifts rapidly with choice of the additional plane wave state and with the size Ω of the box. The energy shift due to the interaction fluctuates correspondingly and there is no definite limit for large Ω for a state of prescribed momentum. It is then useful to study, instead of individual levels, the time dependent propagation of the extra particle or hole.

Denoting by $|\mathbf{0}\rangle$ the ground state of the system without the extra particle then

$$\psi^{+}(\mathbf{r}')|\mathbf{0}\rangle$$

is a (properly antisymmetrized) state in which an extra particle has just been created at the point $\mathbf{r}'$. The subsequent evolution of the system is given by

$$\exp[-iH(t-t')]\psi^{+}(\mathbf{r}')|\mathbf{0}\rangle.$$

The probability amplitude for finding the extra particle at the point $\mathbf{r}$ at time t, with the remainder of the system in its ground state, is

$$\mathbf{G}^{+}(\mathbf{r},t;\,\mathbf{r}',t')=\theta(t-t')\langle\mathbf{0}|\boldsymbol{\psi}(\mathbf{r},t)\boldsymbol{\psi}^{+}(\mathbf{r}',t')|\mathbf{0}\rangle. \quad . \quad . \quad (3.143)$$

Here θ is the usual step function, and

$$\left.\begin{aligned}\boldsymbol{\psi}(\mathbf{r},t&=\exp(iHt)\psi(\mathbf{r})\exp(-iHt),\\ \boldsymbol{\psi}^{+}(\mathbf{r},t)&=\exp(iHt)\psi^{+}(\mathbf{r})\exp(-iHt),\end{aligned}\right\} \quad . \quad . \quad (3.144)$$

are *Heisenberg field operators*. For convenience we are here taking the energy of the state $|\mathbf{0}\rangle$ to be zero. In the same way the propagation of a hole is described by

$$\mathbf{G}^{-}(\mathbf{r},t;\,\mathbf{r}',t)=-\theta(t'-t)\langle\mathbf{0}|\boldsymbol{\psi}^{+}(\mathbf{r}',t')\boldsymbol{\psi}(\mathbf{r},t)|\mathbf{0}\rangle. \quad . \quad (3.145)$$

The conventional 'one-body propagator' is the combination

$$\begin{aligned}\mathbf{G}(\mathbf{r},t;\,\mathbf{r}',t')&=\mathbf{G}^{+}+\mathbf{G}^{-}\\ &=\langle\mathbf{0}|T(\boldsymbol{\psi}(\mathbf{r},t)\boldsymbol{\psi}^{+}(\mathbf{r}',t')|\mathbf{0}\rangle. \quad . \quad . \quad . \quad (3.146)\end{aligned}$$

Let $|\mathbf{N}\rangle$ be the complete set of energy eigenstates for the system with one extra particle, and $|\mathbf{M}\rangle$ the corresponding set for one hole. Then

$$\left.\begin{aligned}\mathbf{G}^{+}(\mathbf{r},t;\,0,0)&=\theta(t)\sum_{\mathbf{N}}\langle\mathbf{0}|\psi(\mathbf{r})|\mathbf{N}\rangle\langle\mathbf{N}|\psi^{+}(0)|\mathbf{0}\rangle\exp(-iE_{\mathbf{N}}t),\\ \mathbf{G}^{-}(\mathbf{r},t;\,0,0)&=-\theta(-t)\sum_{\mathbf{M}}\langle\mathbf{0}|\psi^{+}(0)|\mathbf{M}\rangle\langle\mathbf{M}|\psi(\mathbf{r})|\mathbf{0}\rangle\exp(iE_{\mathbf{M}}t)\end{aligned}\right\} \quad (3.147)$$

where $E_{\mathbf{N}}$ and $E_{\mathbf{M}}$ are the energies of $|\mathbf{N}\rangle$ and $|\mathbf{M}\rangle$ respectively. These have the spatial fourier transforms

$$\left.\begin{aligned}\mathbf{G}_{\mathbf{k}}^{+}(t)&=\theta(t)\sum_{\mathbf{N}}\langle\mathbf{0}|a_{\mathbf{k}}|\mathbf{N}\rangle\langle\mathbf{N}|a_{\mathbf{k}}^{+}|\mathbf{0}\rangle\exp(-iE_{\mathbf{N}}t),\\ \mathbf{G}_{\mathbf{k}}^{-}(t)&=-\theta(-t)\sum_{\mathbf{M}}\langle\mathbf{0}|a_{\mathbf{k}}^{+}|\mathbf{M}\rangle\langle\mathbf{M}|a_{\mathbf{k}}|\mathbf{0}\rangle\exp(iE_{\mathbf{M}}t).\end{aligned}\right\} \quad (3.148)$$

Introducing a small damping factor $\epsilon > 0$ the quantity

$$\mathbf{G}_{\mathbf{k}}^{+}(t)\exp(-\epsilon t)+\mathbf{G}_{\mathbf{k}}^{-}(t)\exp(\epsilon t) \quad . \quad . \quad . \quad (3.149)$$

does not differ appreciably from $\mathbf{G}_{\mathbf{k}}(t)$ for $t \ll \epsilon^{-1}$. It has the Fourier transform in time

$$\mathbf{G}_{\mathbf{k}}(\mathscr{E})=i\sum_{\mathbf{N}}\frac{\langle\mathbf{0}|a_{\mathbf{k}}|\mathbf{N}\rangle\langle\mathbf{N}|a^{+}|\mathbf{0}\rangle}{i\epsilon+\mathscr{E}-E_{\mathbf{N}}}+i\sum_{\mathbf{M}}\frac{\langle\mathbf{0}|a_{\mathbf{k}}^{+}|\mathbf{M}\rangle\langle\mathbf{M}|a_{\mathbf{k}}|\mathbf{0}\rangle}{-i\epsilon+\mathscr{E}-E_{\mathbf{M}}}. \quad . \quad . \quad . \quad (3.150)$$

When $\epsilon \to 0$, for a finite system $\mathbf{G}_{\mathbf{k}}(\mathscr{E})$ has a series of poles, the location of which give energies of the $(A \pm 1)$ particle system with momentum $\pm\mathbf{k}$. If ϵ is kept large compared with the level spacing the discrete structure is averaged away. This happens in effect when in calculations the limit $\Omega \to \infty$ (sums → integrals) is taken before $\epsilon \to 0$. A similar averaging process occurs in the theory of the optical potential in finite nuclei (see, for example, Brown† 1959) to eliminate rapid fluctuations associated with individual compound nucleus states. The resultant propagator still contains statistical information about levels of the $(A \pm 1)$ nuclei. Specifically, in the integral representation

$$\mathbf{G}_{\mathbf{k}}(\mathscr{E})=i\int d\mathscr{E}'\left\{\frac{A_{\mathbf{k}}(\mathscr{E}')}{i\epsilon+\mathscr{E}-\mathscr{E}'}+\frac{B_{\mathbf{k}}(\mathscr{E}')}{-i\epsilon+\mathscr{E}+\mathscr{E}'}\right\} \quad . \quad . \quad (3.151)$$

the positive functions A and B are weighted level densities

$$\left.\begin{aligned}A(\mathscr{E})&=\sum_{\mathbf{N}}\delta(\mathscr{E}-E_{\mathbf{N}})\langle\mathbf{0}|a_{\mathbf{k}}|\mathbf{N}\rangle\langle\mathbf{N}|a_{\mathbf{k}}^{+}|\mathbf{0}\rangle,\\ B(\mathscr{E})&=\sum_{\mathbf{M}}\delta(\mathscr{E}-E_{\mathbf{M}})\langle\mathbf{0}|a_{\mathbf{k}}^{+}|\mathbf{M}\rangle\langle\mathbf{M}|a_{\mathbf{k}}|\mathbf{0}\rangle,\end{aligned}\right\} \quad (3.152)$$

where the 'δ'-functions are supposed of width sufficient to cover many levels.

The propagator $\mathbf{G}$ can be represented by a linked diagram expansion of the type already used. We again need the relation (3.3) between real and model states

$$|\mathbf{0}\rangle=\frac{S(0,-\infty)|0\rangle}{\langle\mathbf{0}|S(0,-\infty)|0\rangle}$$

and the similar formula

$$\langle\mathbf{0}|=\frac{\langle 0|S(\infty,0)}{\langle 0|S(\infty,0\rangle|\mathbf{0}\rangle}.$$

The Heisenberg operators are given in terms of interaction representation operators by

$$\left.\begin{aligned}\boldsymbol{\psi}(\mathbf{r},t)&=S^{-1}(t,0)\,\psi(\mathbf{r},t)\,S(t,0),\\ \boldsymbol{\psi}^{+}(\mathbf{r},t)&=S^{-1}(t,0)\,\psi^{+}(\mathbf{r},t)\,S(t,0).\end{aligned}\right\} \quad . \quad . \quad . \quad (3.153)$$

Thus

$$\mathbf{G}^{+}(\mathbf{r},t;\mathbf{r}',t')=\frac{\langle 0|S(\infty,t)\psi(\mathbf{r},t)S(t,t')\psi^{+}(\mathbf{r}',t')S(t',-\infty)|0\rangle}{\langle 0|S(\infty,-\infty)|0\rangle} \quad . \quad . \quad . \quad (3.154)$$

---

† Brown, G. E., *Rev. mod. Phys.*, **31**, 893.

and likewise for G^-. The S's are replaced by perturbation series like (3.7), with the result

$$\mathbf{G}(\mathbf{r},t\,;\,\mathbf{r}',t') = \frac{\langle 0|\sum \frac{(-i)^n}{n!}\int dt_1\ldots dt_n T(\psi(\mathbf{r},t)\,\psi^+(\mathbf{r}',t')V(t_1)\ldots V(t_n))|0\rangle}{\langle 0|\sum \frac{(-i)^n}{n!} dt_1\ldots dt_n\,T\,(V(t_1)\ldots V(t_n))|0\rangle}. \quad . \quad . \quad . \quad (3.155)$$

This differs from (3.8) in the appearance of ψ and ψ^+ in the numerator rather than $V(0)$, and by time integrals from $-\infty$ to $+\infty$ rather than from $-\infty$ to 0. Evaluation by Wick's theorem leads to diagrammatic representation as before. The numerator diagrams now have external lines terminating at points ($\mathbf{r}$, t) and ($\mathbf{r}'$, t'); these represent factor pairings involving the extra ψ and ψ^+. Numerator diagrams with parts not connected to the external lines precisely cancel against the denominator, leaving a linked diagram expansion for $\mathbf{G}$.

The diagrams for $\mathbf{G}$ are exactly the same as those for G in § 3.7. The only difference is in the extended range of the associated time integrals. The quantity G, for negative t and t', is actually given by

$$G(\mathbf{r},t\,;\,\mathbf{r}',t') = \frac{\langle 0\,|\,T(\psi(\mathbf{r},t),\psi^+(\mathbf{r}',t'))|\mathbf{0}\rangle}{\langle 0\,|\mathbf{0}\rangle} \quad . \quad . \quad (3.156)$$

The appearance of the model state $\langle 0\,|$ on the left replaces the quantity $S(\infty, t)$ of (3.154) by $S(0, t)$, and so restricts the time integrations as noted.

We again obtain equations like (3.97)

$$\begin{aligned}\mathbf{G} &= G^0 - iG^0\,\mathbf{M}\,\mathbf{G}\\ &= G^0 - i\,\mathbf{G}\,\mathbf{M}\,G^0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.157)\end{aligned}$$

where $\mathbf{M}$, like M, arises from proper self-energy insertions only. Unlike G and M, $\mathbf{G}$ and $\mathbf{M}$ are translationally invariant. They have therefore Fourier decompositions of the type†

$$\mathbf{G}(\mathbf{r},t\,;\,\mathbf{r}',t') = \int\frac{d\mathscr{E}}{2\pi}\sum \mathbf{G}_{\mathbf{k}}(\mathscr{E})\exp\,[i\mathbf{k}\,.\,(\mathbf{r}-\mathbf{r}') - i\mathscr{E}(t-t')].$$

From (3.157)

$$\mathbf{G}_{\mathbf{k}}(\mathscr{E}) = G_k{}^0(\mathscr{E})\{1 + \mathbf{M}_k(\mathscr{E})\,\mathbf{G}_k(\mathscr{E})\}^{-1}$$

or

$$\mathbf{G}_{\mathbf{k}}(\mathscr{E}) = \left\{\mathscr{E} - \left(\frac{k^2}{2M} \mp i\epsilon\right) - \mathbf{M}_k(\mathscr{E})\right\}^{-1}. \quad . \quad . \quad (3.158)$$

Now in the first order of perturbation theory $\mathbf{M}_k$ reduces to the usual Hartree–Fock potential, which depends on k but not $\mathscr{E}$. Equation (3.158) then shows that the propagation is modified only in that the velocity dependent effective potential is added to the kinetic energy. More generally $\mathbf{M}$ can still be regarded as the effective single body potential but (because of the $\mathscr{E}$ dependence of its transform) it is an integral operator (non-local) with respect to time as well as space. In the theory of the optical model of finite nuclei (Bell and Squires 1959, Bell 1960) a quantity analogous to $\mathbf{M}$ is shown to give correctly the elastic scattering amplitude. The construction of this quantity is formally identical with that of $\mathbf{M}$ and can plausibly be approximated by the latter in the central region of a large nucleus.

---

† In order that Fourier transforms with respect to time exist in an unsophisticated sense, we again introduce some artificial damping. In perturbation theory this is conveniently done by adding $(-i\epsilon)$ to the kinetic energy of states above the fermi surface, and $+i\epsilon$ to that of states below. This corresponds to the modification (3·148) applied to the unperturbed propagator G^0. The damping of $\mathbf{G}$ and $\mathbf{M}$, constructed by perturbation series involving G^0, is then more complicated.

A natural generalization of the single particle energy of Hartree–Fock theory is given by the root of the equation

$$e_k' = \frac{k^2}{2M} - \mathbf{M}_k(e_k'). \qquad \ldots\ldots\ldots (3.159)$$

This must be solved by iteration to yield an explicit series. The quantity e_k' is in general complex, indicating the transient nature of the particle or hole motion, which dissipates into more complex excitations. In the first order in v or in the K matrix, e_k' of (3.159) is the same as the single particle energy of Brueckner theory § 3.7. But conceptually it is different, for it refers to real rather than virtual particles and holes; moreover it has meaning to all orders of perturbation theory, and not just in a particular approximation scheme.

In the particular case $k=k_f$ the single particle energy is real and equal to the separation energy e_f' of § 3.9. This is the theorem of Hugenholtz and Van Hove (1958). To see it, consider for a moment the case of finite Ω. Then the propagator $\mathbf{G}_{kf}(\mathscr{E})$ has a discrete set of poles whose positions give the energies of the $(A \pm 1)$ particle systems. Constructing from (3.159) the series for the energy level continuous with $e_f' = k_f^2/2M$ in perturbation theory, setting $\epsilon = 0$ and then letting $\Omega \rightarrow \infty$, we must obtain a series for the quantity e_f'. Moreover, this series is essentially the derivative with respect to k_f of the energy per particle (§ 3.9). Now the series obtained from (3.159) for the optical model energy e_k' in the limit $k \rightarrow k_f$ can differ from the series just described only in the presence of the infinitesimal imaginary parts $i\epsilon$ in energy denominators (for in this case the limit $\Omega \rightarrow \infty$ is taken first). Thus e_k' as $k \rightarrow k_f$ is the derivature of a series which is obtained from that for E/A by inserting such imaginary quantities. However, because vanishing energy denominators do not occur there with finite weight, this makes no difference when the limit $\epsilon \rightarrow 0$ is taken. Thus $e_k' \rightarrow e_f'$ as $k \rightarrow k_f$.

This provides, in conjunction with (3.137) another way of calculating the total energy per particle. It was so used for example by Galitskii (1958). There are other ways of obtaining the energy from the one-body propagator (3.146) or from the similarly defined two-body propagator, e.g. Klein and Prange (1958), Galitskii (1958). Such formalisms have the advantage, over that developed in § 3.3, of not introducing explicitly an origin of time and so working always with translationally invariant quantities. On the other hand, the total energy shift is not given directly but only after for example an additional integration over k_f, or over a coupling constant, or after a separate calculation of kinetic and potential energies.

3.11. *Final Remarks*

We have freely made rearrangements in the diagram expansion. In the next chapter it will be seen that summations over certain classes of diagram diverge for arbitrarily small interaction strength g for sufficiently large volume Ω. The complete series may itself have this character, although that has not been proved. Despite this a rearranged expansion might remain valid. Suppose for example that (for fixed large Ω) the rearrangement could be justified for small enough g, and that the ground-state energy varies analytically as g is increased to the required value. Then if the rearranged series also gives a result analytic in g it is the correct result. However, the ground-state energy may not vary smoothly with g; for example, some originally higher level may come down and take over from what was originally the ground state. The considerations of § 4 suggest that this may in fact occur for nuclear matter. Although the finite temperature version of the formalism might then remain applicable (§ 4.4) the Brueckner–Goldstone approach of this chapter would no longer answer the right question. Indeed the particular rearrangements used may not provide a sensible answer to any question, for we will see that formidable difficulties present themselves in connection with them.

§ 4. SINGULARITIES AND SUPERFLUIDITY

4.1. *Introduction*

The manipulations of the previous chapter have been made without regard to whether the series considered are suitably convergent, or whether the integral equations obtained actually have non-singular solutions. In fact, however, the Brueckner K matrix, expressed as a series in V, is divergent in an infinite system however weak the potential. Further, for nuclear matter there is good reason to believe that accurate evaluation of the results of the theory would lead to a singular expression for the energy. The finiteness of the results obtained in the previous chapter must then be attributed to some of the cruder approximations of the working (e.g. the coarse mesh used in the numerical integrations and the treatment of 'off energy shell' effects in the Brueckner–Gammel calculations, and the use of only second order in the long-range part of the potential in the work of Moszkowski and Scott). These matters are discussed in § 4.2.

In seeking some understanding of the singularity, we discuss in § 4.3 the effect of including additional diagrams in K, and in § 4.4 the generalization of the theory to non-zero temperatures. Finally, in the last four sections is considered an entirely different, essentially non-perturbative, approach to the theory, which leads to the suggestion that nuclear matter is superfluid. The basic theory for zero temperature is outlined in § 4.5, and for finite temperature in § 4.6. Some possible corrections are discussed in § 4.7, and § 4.8 contains explicit application to nuclear matter.

4.2. *The K-matrix Singularity*

To illustrate this singularity (Cooper 1956 is the first published reference†) we use a separable potential, which we take to have the form

$$\langle \mathbf{k}' | v | \mathbf{k} \rangle = \Omega^{-1} g \, v_{k'} v_k \qquad \ldots \ldots \ldots \quad (4.1)$$

where $\mathbf{k} = \frac{1}{2}(\mathbf{k}_1 - \mathbf{k}_2)$ and $\mathbf{k}' = \frac{1}{2}(\mathbf{k}_1{}' - \mathbf{k}_2{}')$, $\mathbf{k}_1$ and $\mathbf{k}_2$ being the initial momenta of the interacting particles and $\mathbf{k}_1{}'$ and $\mathbf{k}_2{}'$ their final momenta. We assume v_k to be real and positive. Such a potential acts only in s states and so gives saturation if g is negative. The K matrix, in the effective field approximation of § 3.7, for nucleons of total momentum $\mathbf{p}$, is given from (3.17) by

$$\begin{aligned} \langle \mathbf{k}' | K(\mathbf{p}) | \mathbf{k} \rangle &= \Omega^{-1} g \, v_{k'} v_k \\ &\quad + \Omega^{-1} g \sum_{\mathbf{k}''} \frac{v_{k'} v_{k''} f(\mathbf{k}'', \mathbf{p}) \langle \mathbf{k}'' | K(\mathbf{p}) | \mathbf{k} \rangle}{e(\mathbf{k} + \frac{1}{2}\mathbf{p}) + e(\mathbf{k} - \frac{1}{2}\mathbf{p}) - e(\mathbf{k}'' + \frac{1}{2}p) - e(\mathbf{k}'' - \frac{1}{2}\mathbf{p})} \end{aligned} \qquad (4.2)$$

where f is defined in (3.60). The parameter $\mathscr{E}$ has been given the 'on the energy shell' value $e(\mathbf{k} + \frac{1}{2}\mathbf{p}) + e(\mathbf{k} - \frac{1}{2}\mathbf{p})$ since this is what is required in (3.122). We have used the notation $e(\mathbf{k})$ for e_k. Equation (4.2) is readily solved to give

$$\langle \mathbf{k}' | K(\mathbf{p}) | \mathbf{k} \rangle = \frac{\Omega^{-1} g \, v_{k'} v_k}{1 + g \sum(\mathbf{k}, \mathbf{p})} \qquad \ldots \ldots \quad (4.3)$$

where

$$\sum(\mathbf{k}, \mathbf{p}) = \frac{1}{\Omega} \sum_{\mathbf{k}''} \frac{v_{k''}{}^2 f(k'', p)}{e(\mathbf{k}'' + \frac{1}{2}\mathbf{p}) + e(\mathbf{k}'' - \frac{1}{2}\mathbf{p}) - e(\mathbf{k} + \frac{1}{2}\mathbf{p}) - e(\mathbf{k} - \frac{1}{2}\mathbf{p})}$$

or, for $\Omega \to \infty$,

$$\sum(\mathbf{k}, \mathbf{p}) = \int_{|\mathbf{k}'' \pm \frac{1}{2}\mathbf{p}| > k_f} \frac{d\mathbf{k}''}{(2\pi)^3} \frac{v_k{}''^2}{e(\mathbf{k}'' + \frac{1}{2}\mathbf{p}) + e(\mathbf{k}'' - \frac{1}{2}\mathbf{p}) - e(\mathbf{k} + \frac{1}{2}\mathbf{p}) - e(\mathbf{k} - \frac{1}{2}\mathbf{p})} \, . \qquad \ldots \quad (4.4)$$

If $e(l) < e(m)$ for all $l(> k_f)$ and all $m(< k_f)$, then the integrand in (4.4) is positive. We can then obtain a lower bound for $\sum$ by replacing the condition $k_f < |\mathbf{k}'' \pm \frac{1}{2}\mathbf{p}|$ by the stronger one

$$k_f < k'' \pm \tfrac{1}{2}p < k_f + \lambda$$

where λ is any positive quantity (which we later take less than k_f). For a reasonable form for $e(k)$ we have

$$\frac{4\pi k''^2}{(2\pi)^3} \frac{(k'' - k) \, v_{k''}{}^2}{e(\mathbf{k}'' + \frac{1}{2}\mathbf{p}) + e(\mathbf{k}'' - \frac{1}{2}\mathbf{p}) - e(\mathbf{k} + \frac{1}{2}\mathbf{p}) - e(\mathbf{k} - \frac{1}{2}\mathbf{p})} < A \qquad (4.5)$$

for all $\mathbf{k}''$ in the above range and for $|\mathbf{k} \pm \frac{1}{2}\mathbf{p}| < k_f$, where A is some positive number. Then

$$\sum > A \int_{k_f + \frac{1}{2}p}^{k_f - \frac{1}{2}p + \lambda} \frac{dk''}{k'' - k} = A \log \frac{k_f - \frac{1}{2}p + \lambda - k}{k_f + \frac{1}{2}p - k} \qquad \ldots \quad (4.6)$$

whence $\sum \to \infty$ as $k \to k_f$, $p \to 0$.

---

† Van Hove (1959) gives many references.

As a first consequence of this note that, whatever the sign and magnitude of g, there is a region where $\sum > 1$; there the series expansion of K in powers of g is divergent. Moreover the associated power series for the energy shift is also divergent. This is of the form

$$\frac{\Delta E}{\Omega} = g \sum (-g)^n \Delta n \qquad \text{. . . . . .} \qquad (4.7)$$

where

$$\Delta_n = 12 \int_{|\mathbf{k} \pm \frac{1}{2} p| < k_f} \frac{d\mathbf{k}}{(2\pi)^3} \frac{d\mathbf{p}}{(2\pi)^3} v_k{}^2 \{ \textstyle\sum(\mathbf{k}, \mathbf{p}) \}^n. \qquad \text{. .} \qquad (4.8)$$

Imposing the more restrictive condition

$$k_f - \lambda < k \pm \tfrac{1}{2} p < k_f$$

and using (4.6), we find

$$\Delta_n > B P_n$$

where B is a positive number independent of n, and

$$P_n = \lambda^{-4} \int_0^\lambda d_p\, p^2 \int_{k_f + \frac{1}{2}p - \lambda}^{k_f - \frac{1}{2}p} dk \left[\log \frac{\lambda}{k_f + \frac{1}{2}p - k} \right]^n .$$

By change of variables and partial integration

$$\begin{aligned} P_n &= \int_0^1 dt\, t^2 \int_t^1 dx \left(\log \frac{1}{x} \right)^n \\ &= n P_{n-1} - \int_0^1 dt\, t^3 \left(\log \frac{1}{t} \right)^n \\ &= n P_{n-1} - n!/4^{n+1} \end{aligned}$$

whence

$$P_n = n!\,(P_0 - 1/12) + \tfrac{1}{3} \frac{n!}{4^{n+1}} ;$$

so that, for large n, $P_n/P_{n-1} \rightarrow n$, and

$$\Delta_n / \Delta_{n-1} \geqslant n.$$

The series (4.7) thus diverges for all g.

The divergence of the power series, even for arbitrarily weak non-singular potentials, is unexpected. But it is not a vital matter for Bruckner theory, whose final equations make no reference to such an expansion. A much more alarming consequence of (4.7) is that for $g < 0$ the denominator in (4.3) can vanish for values of $\mathbf{k}$ and $\mathbf{p}$ required in the evaluation of ΔE. Such a zero occurs unless g is larger than the maximum of $\sum^{-1}$. The Brueckner expression for ΔE is then a meaningless singular integral. For a large but finite system the singularity may not occur for the allowed discrete values of $\mathbf{k}$ and $\mathbf{p}$, depending critically on the value of g and the size Ω of the system; as Ω or g are varied the energy shift ΔE oscillates infinitely.

The case of more general potentials† has been investigated by Emery (1959, 1960) again using the effective field approximation and considering only $p=0$. The 'on energy shell' K-matrix equation becomes

$$K|\mathbf{m}\rangle = v|\mathbf{m}\rangle + \sum_{\mathbf{l}} \frac{v|\mathbf{l}\rangle\langle\mathbf{l}|K|\mathbf{m}\rangle}{2e(m)-2e(l)}. \qquad (4.9)$$

Emery studied the homogeneous eigenvalue equation corresponding to (4.9)

$$K|\mathbf{m}\rangle = \frac{\mu}{2}\sum_{\mathbf{l}} \frac{v|\mathbf{l}\rangle\langle\mathbf{l}|K|\mathbf{m}\rangle}{e(m)-e(l)}. \qquad (4.10)$$

A necessary and sufficient condition for the series expansion of K, in powers of v, to exist is that none of the eigenvalues μ are less than 1 in magnitude. Further, (4.9) has a non-singular solution if, and only if, there is no eigenvalue of (4.10) equal to unity. Emery (1959) showed that as $m \to k_f$ there is always one eigenvalue that tends to zero, thus demonstrating that there is a range of $\mathbf{m}$ for which the series expansion of K is divergent. He also showed that with the central s-wave part of the Brueckner–Gammel potential (§ 1.3), and with the effective mass approximation (§ 3.7) for the $e(k)$, there is not a non-singular solution of (4.9) in the triplet state if $M^* > 0{\cdot}75\,M$ nor in the singlet state if $M^* > 0{\cdot}56\,M$. In the calculations of Brueckner and Gammel the effective mass at the fermi surface

$$M^{*-1} = \frac{1}{k_f}\left.\frac{\partial e(k)}{\partial k}\right|_{k_f}$$

was found to be $0{\cdot}73\,M$.

Emery's result for the singlet s-state was confirmed by Nuttall *et al.*‡ (1958, unpublished) who calculated the K matrix directly from the equations used by Brueckner and Gammel (§ 3.8). With Δ (see below (3.128)) chosen so that $\bar{e}(2e_m, l)$ was continuous with e_m as m and l tend to k_f (this makes the energy denominators come to zero, as in the effective field approximation) they found that a singularity occurred in $\langle\mathbf{m}|K|\mathbf{m}\rangle$ for m equal to $1{\cdot}486 f^{-1}$ (with $k_f = 1{\cdot}5 f^{-1}$). When Δ was increased the particle energies $\bar{e}(2e_f, l)$ just above the fermi surface increased so that there was an energy gap in the spectrum. The singularity disappeared when this gap became equal to about 0·4 MeV.

In the calculations of Brueckner and Gammel the singularity would not have been noticed because of the coarseness of the mesh used in the numerical integrations. Also the value of Δ used in the calculations was probably such as to prevent the singularity occurring. The removal of the singularity by this crude treatment of 'off energy shell' effects is of course unsatisfactory. Without this approximation there is continuity in e_m and $e_l(l, 2e_m)$ at the fermi surface.

The question arises as to whether these peculiar features of Brueckner theory imply some peculiarity of the system, or merely show that an unfortunate selection of terms has been made. For example, is the complete perturbation series (as distinct from the Brueckner series) always divergent? That this need not be so has been illustrated by Bell (1959) in the many-body problem with one-body forces. There one supposes the fermions to interact not with one another but with an external potential. A K-matrix approximation to the energy again leads to a singular integral, for a weak separable attraction, and a divergent series. However, this simple problem can easily be solved exactly (Fukuda and Newton 1956, de Witt 1956); the energy shift is an integral of an expression linear in the scattering phase shifts, having a convergent power series for sufficiently weak interaction. It has by no means been proved that this is not the case for the two-body force also. However, as we shall see in § 4.3, extra terms corresponding to those which remove the singularity in the one-body force problem do not do so in the two-body force problem. It is now generally believed that the divergence of the K matrix is associated with the inadequacy of the unperturbed fermi gas as a point of departure, and to the existence of a 'superfluid' state (§ 4.5).

---

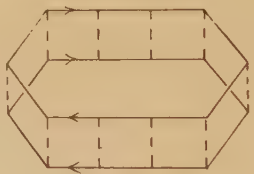
† See also Van Hove (1959, 1960).

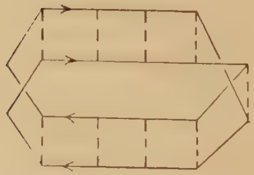
‡ Clement, C., Nuttall, J., and Sampanthar, S

4.3. *The Inclusion of Additional Terms in the Perturbation Series*

In § 3.4 we saw that it was possible to express the sum of a certain class of diagrams, the particle–particle interactions, as the solution of an integral equation, and in § 3.7 we saw that certain additional diagrams could be included by modifying the energy denominators. We shall now show how to include still further diagrams which restore symmetry with respect to the fermi surface. Since the singularity discussed in § 4.2 is associated with the interaction of particles and holes near the fermi surface, this might be expected to provide some insight into the problem.

The simplest extension of § 3.4 is to include all hole–hole interactions as well as all particle–particle interactions, i.e. to sum all time ordered diagrams of the types


. . . . (4.11)


. . . . (4.11′)

with an arbitrary number of particle–particle and hole–hole interactions.

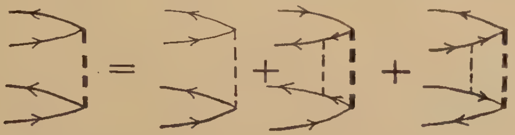
This can be done (Chisholm and Squires 1958) by replacing (3.39) by the two equations

$$\langle \mathbf{m}_1\mathbf{m}_2|\bar{K}|\mathbf{m}_1\mathbf{m}_2\rangle = \langle \mathbf{m}_1\mathbf{m}_2|v|\mathbf{m}_1\mathbf{m}_2\rangle + \sum_{\mathbf{l}_1,\,\mathbf{l}_2} \frac{\langle \mathbf{m}_1\mathbf{m}_2|v|\mathbf{l}_1\mathbf{l}_2\rangle\langle \mathbf{l}_1\mathbf{l}_2|\bar{K}|\mathbf{m}_1\mathbf{m}_2\rangle}{(m_1{}^2+m_2{}^2-l_1{}^2-l_2{}^2)/2M} \quad . \; . \; . \; . \quad (4.12)$$

and

$$\begin{aligned}\langle \mathbf{l}_1\mathbf{l}_2|\bar{K}|\mathbf{m}_1\mathbf{m}_2\rangle = \langle \mathbf{l}_1\mathbf{l}_2|v|\mathbf{m}_1\mathbf{m}_2\rangle &+ \sum_{\mathbf{l}_1,\,\mathbf{l}_2} \frac{\langle \mathbf{l}_1\mathbf{l}_2|v|\mathbf{l}_1'\mathbf{l}_2'\rangle\langle \mathbf{l}_1'\mathbf{l}_2'|\bar{K}|\mathbf{m}_1\mathbf{m}_2\rangle}{(m_1{}^2+m_2{}^2-l_1'^2-l_2'^2)/2M}\\ &+ \sum_{\mathbf{m}_1,\,\mathbf{m}_2} \frac{\langle \mathbf{l}_1\mathbf{l}_2|\bar{K}|\mathbf{m}_1'\mathbf{m}_2'\rangle\langle \mathbf{m}_1'\mathbf{m}_2'|v|\mathbf{m}_1\mathbf{m}_2\rangle}{(m_1'^2+m_2'^2-l_1{}^2-l_2{}^2)/2M}. \quad (4.13)\end{aligned}$$

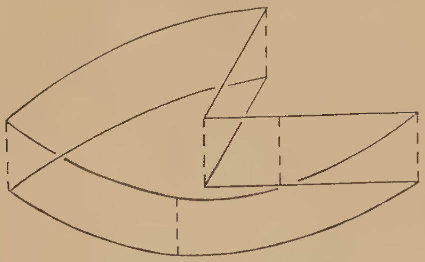
If we denote a $\bar{K}$ interaction by a heavy dotted line in a time ordered diagram, this last equation can be written diagramatically as



. . . (4.14)

which can obviously be iterated to give the desired result. For a separable attractive potential the additional terms included have the same sign as the particle–particle interactions of the Brueckner K matrix and consequently the power series is still divergent. With the same potential it was shown by Chisholm and Squires (1959) and Mehta (1959) that for a finite system the energy obtained from (4.12), (4.13) and (3.41)—with K replaced by $\bar{K}$—again oscillates infinitely as the volume is varied (for sufficiently large volume) and has no limit as $\Omega \to \infty$. Thus, although an unambiguous solution, containing a δ-function, exists for the integral equation obtained by taking the infinite limit in (4.13), it is not possible to attach any meaning to the energy resulting from this 'solution'.

A further extension can be made by including in the sum all 'pair production' diagrams, e.g.



. . . (4.15)

Indeed, if we use Feynmann diagrams (§ 3.3) these are not distinguished from the particle–particle and hole–hole interaction diagrams, since the order of the interactions is not significant. Mehta (1960) calls all diagrams including hole–hole, particle–particle and pair production interactions 'generalized ladder diagrams'. In § 3.7 we saw that the contribution of all generalized ladder diagrams could be obtained by defining the operator K' by

$$K' = v - iK'GGv \qquad (3.100)$$

where we are using the matrix notation of § 3.7. For G we shall use the effective field approximation (3.119); then

$$\begin{aligned} G_k(t,t') &= \exp[-ie(k)(t-t')], \quad k>k_f \text{ and } t>t' \\ &= -\exp[-ie(k)(t-t')], \quad k<k_f \text{ and } t<t' \\ &= 0 \text{ otherwise.} \end{aligned} \qquad (4.16)$$

We can then define

$$K'(\mathscr{E}) = \int_{-\infty}^{0} \exp(-i\mathscr{E}t)\, K'(0,t)\, dt \qquad (4.17)$$

and, from (3.100), (4.16) and (4.17), obtain

$$\begin{aligned} K'(q) = v - \sum_{\mathbf{l}_1,\mathbf{l}_2} \frac{K'(\mathscr{E})|\mathbf{l}_1\mathbf{l}_2\rangle\langle\mathbf{l}_1\mathbf{l}_2|v}{e(l_1)-e(l_2)-\mathscr{E}} \\ - \sum_{\mathbf{m}_1,\mathbf{m}_2} \frac{[K'(e_{m_1}=e_{m_2})-K'(\mathscr{E})]|\mathbf{m}_1\mathbf{m}_2\rangle\langle\mathbf{m}_1\mathbf{m}_2|v}{e(m_1)+e(m_2)-\mathscr{E}}. \end{aligned} \qquad (4.18)$$

The energy shift is given, from (3.105), by

$$\Delta E = \tfrac{1}{2} \sum_{\mathbf{m}_1,\mathbf{m}_2} \langle\mathbf{m}_1\mathbf{m}_2|K'\left(\frac{m_1{}^2+m_2{}^2}{2M}\right)|\mathbf{m}_1\mathbf{m}_2\rangle. \qquad (4.19)$$

Equation (4.18) was obtained by Prange (1960). The contribution of the generalized ladder diagrams to the energy had previously been obtained, in a different form, by Galitskii (1958). A very simple method of obtaining the above result, due to Mehta (1960), is given below.

Prange (1960) solved the integral eqn. (4.18) for a pure hard core potential in nuclear matter and found that the effect of the additional terms increased the energy shift by 10% over that found by the Brueckner method. This is a surprisingly large correction to arise from the short-range hard-core potential, but confirms the result of § 3.5 that the low density expansion for the pure hard core converges slowly. No treatment of the above integral equation for K' with a realistic potential is available, but the estimate of the hole-hole scattering correction in lowest order discussed in § 3.6, suggests that some cancellation occurs and that it is actually much smaller than with the hard core alone.

However, for an attractive separable potential, the additional terms included in K' have the same sign as those already included in Brueckner's K, so the series for K' is certainly divergent whenever the series for K is divergent. We can show further that, for such a potential, the energy obtained from (4.18) for a finite system becomes complex as the volume of the system becomes sufficiently large. It is clear, then, that the introduction of generalized ladders† does little to resolve the difficulties discussed in § 4.2.

To obtain the above result, note that with the potential (4.1) the solution to (4.18) can be obtained in the form

$$\langle \mathbf{k}' | K'(\mathscr{E}) | \mathbf{k} \rangle = \Omega^{-1} g \, v_{k'} v_k \mathscr{F}(\mathscr{E}) \quad . \quad . \quad . \quad . \quad (4.20)$$

where, for $\mathbf{p}=0$,

$$\mathscr{F}(\mathscr{E}) = 1 - \frac{g}{\Omega} \sum_{\mathbf{l}} \frac{v_l^2 \mathscr{F}(\mathscr{E})}{2e_l - \mathscr{E}} - \frac{g}{\Omega} \sum_{\mathbf{m}} \frac{v_m^2 [\mathscr{F}(2e_m) - \mathscr{F}(\mathscr{E})]}{2e_m - \mathscr{E}} \qquad (4.21)$$

or

$$\mathscr{F}(\mathscr{E}) = \left[1 + \frac{g}{\Omega} \sum_{\mathbf{l}} \frac{v_l^2}{2e_l - \mathscr{E}} - \frac{g}{\Omega} \sum_{\mathbf{m}} \frac{v_m^2}{2e_m - \mathscr{E}} \right]^{-1} \left(1 - \frac{g}{\Omega} \sum \frac{v_m^2 \mathscr{F}(2e_m)}{2e_m - \mathscr{E}} \right). \quad . \quad . \quad . \quad (4.22)$$

Now this does not determine $\mathscr{F}(\mathscr{E})$, for clearly the quantities $\mathscr{F}(2e_m)$ can be specified arbitrarily. To complete the determination we specify that the solution be continuous with the iterative solution valid for small g (the manner of derivation obviously requires this solution). Now, from (4.21), one sees that the iterative solution has, for small g, singularities near $\mathscr{E} = 2e_l$ but not near $\mathscr{E} = 2e_m$. Suppose the quantities λ_i are those zeros of

$$Z(\mathscr{E}) = 1 + \frac{g}{\Omega} \sum_{\mathbf{l}} \frac{v_l^2}{2e_l - \mathscr{E}} - \frac{g}{\Omega} \sum_{\mathbf{m}} \frac{v_m^2}{2e_m - \mathscr{E}} \quad . \quad . \quad . \quad (4.23)$$

each of which approaches an e_m as $g \to 0$. Then, for $\mathscr{F}(\lambda_i)$ to be finite, we need, from (4.22),

$$1 - \frac{g}{\Omega} \sum_{\mathbf{m}} \frac{v_m^2 \mathscr{F}(2e_m)}{2e_m - \lambda_i} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.24)$$

for all i. This determines the $\mathscr{F}(2e_m)$. To find the energy shift we multiply (4.24) by the common denominator and regard the result as a polynomial equation for the λ_i. The sum of its roots is then given by

$$\sum_i \lambda_i = \sum_{\mathbf{m}} 2e_m - \frac{g}{\Omega} \sum_{\mathbf{m}} v_m^2 \mathscr{F}(2e_m). \quad . \quad . \quad . \quad . \quad (4.25)$$

From (4.19), (4.20) and (4.25) the $\mathbf{p}=0$ contribution to the energy shift is, ignoring trivial factors,

$$\Delta E_{\mathbf{p}=0} = \sum_{\mathbf{m}} \langle \mathbf{m} | K(2e_m) | \mathbf{m} \rangle$$

$$= \sum_{\mathbf{m}} (2e_m - \lambda_m). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.26)$$

---

† The difficulties with the generalized ladders appear to have been observed first by J. Goldstone (unpublished).

This gives a reasonable result for sufficiently small $|g|$ and Ω. However, it is easily seen that on increasing $|g|$ (with $g<0$) or Ω, two of the roots of $Z(\mathscr{E})=0$, including the largest of the λ_i, become complex. It can easily be shown that the imaginary part of the largest λ_i becomes independent of Ω as $\Omega \to \infty$, so that, since the other λ_i are all $0(1/\Omega)$, the imaginary part of the energy shift (4.26) is of the same order in Ω as the real part.

Mehta (1960) has given an instructive alternative derivation of the above results, exploiting the triviality of the one-body force problem. Consider such a problem with occupied levels of energy $2e_m$, unoccupied levels with energy $2e_l$, and an external perturbaing potential with matrix elements given by (4.1). The ground state energy shift is

$$\Delta E = \sum_{m} (\lambda_m - 2e_m) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.27)$$

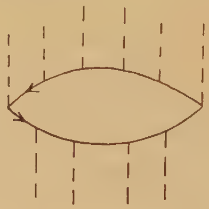
where the λ's are the lowest eigenvalues of the one-body Schrödinger equation

$$\sum_{\mathbf{k}'} \{2e_k \delta_{\mathbf{kk}'} + \langle \mathbf{k} | v | \mathbf{k}' \rangle\} \langle \mathbf{k}' | \psi \rangle = \lambda \langle \mathbf{k} | \psi \rangle. \quad . \quad . \quad . \quad (4.28)$$

These eigenvalues are in fact the roots of

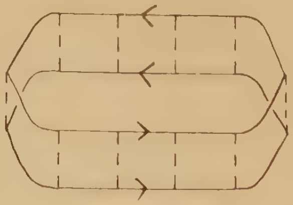
$$1 + \frac{g}{\Omega} \sum_{\mathbf{k}} \frac{v_k{}^2}{2e_k - \mathscr{E}} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.29)$$

Now, in terms of perturbation diagrams the complete solution of this problem is given by the analogues of the generalized ladders (there are no others). A typical such diagram is



. . . . . . (4.30)

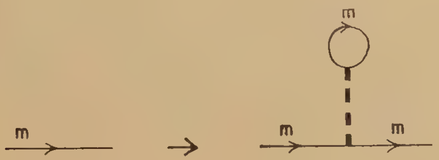
The sum of all such diagrams must give (4.27) and (4.29) exactly, as can be verified. The two-body force diagram analogous to (4.30) is



. . . . . (4.31)

To obtain the contribution arising from (4.31) from that arising from (4.30) we recall the rules of § 3.3 and introduce (i) a factor (-1) to account for the extra closed loop and (ii) a factor (-1) for each backward line in (4.30), to account for the fact that they appear twice in (4.31). The latter can be effected simply by replacing v_m^2 by $-v_m^2$ in all expressions. Thus, from (4.27) and (4.29) we immediately obtain the result (4.23) and (4.26). It will be noted that the difficulty of complex eigenvalues never occurs in (4.29), and this is an essential difference between the one and two-body force problems.

Finally in this section we recall that, in the derivation of the Brueckner approximation starting from a wavefunction with two-body correlations (§ 2.5) we obtained an expression which included the energy shift $\langle \mathsf{m} | K | \mathsf{m} \rangle$ in the energy denominators (see (2.43)). This could be obtained from the low density approximation of § 3.4 by modifying the hole propagators to include the interaction of the hole with itself, i.e.



. . . (4.32)

Gottfried (1958) considered the general solution of this modified K matrix equation for the separable potential. For simplicity we take total momentum zero and put $\langle \mathsf{m} | K | \mathsf{m} \rangle = \Delta_m$ whence

$$\begin{aligned}\Delta_m &= \frac{g}{\Omega} v_m^2 - \frac{g}{\Omega} \sum_{\mathsf{l}} \frac{v_l^2 \Delta_m}{e_l - e_m - \Delta_m} \\ &= \frac{g}{\Omega} v_m^2 - \frac{g}{\Omega} \sum_{\mathsf{l}} \frac{v_l^2 \Delta_m}{e_l - e_m} - \Delta_m^2 \frac{g}{\Omega} \sum_{\mathsf{l}} \frac{v_l^2}{(e_l - e_m)(e_l - e_m - \Delta_m)}. \qquad (4.33)\end{aligned}$$

If the resulting equation has a solution with $\Delta_m \to 0$ as $\Omega \to \infty$ it is reasonable to neglect the Δ_m in the denominator of the last term of (4.33). The quadratic equation for Δ_m then has solutions

$$\Delta_m = -\frac{B_m \pm \left[B_m^2 + 4 \dfrac{g^2}{\Omega} A_m v_m^2 \right]^{1/2}}{2gA_m} \qquad . . . (4.34)$$

with

$$A_m = \frac{1}{\Omega} \sum_{\mathsf{l}} \frac{v_l^2}{(e_l - e_m)^2}, \qquad . . . . (4.35)$$

$$B_m = 1 + \frac{g}{\Omega} \sum_{\mathsf{l}} \frac{v_l^2}{e_l - e_m}. \qquad . . . . (4.36)$$

Away from the neighbourhood of the zero in B the second term under the square root of (4.34) is small compared to the first for large Ω and we can expand, thus obtaining the usual solution if we take the negative root, i.e.

$$\Delta_m = \frac{g v_m^2}{\Omega B_m}. \qquad . . . . (4.37)$$

However, near $B_m = 0$, the solution of (4.34) does not have a pole but is given by

$$\Delta_m \simeq \frac{v_m}{\Omega^{1/2} A_m^{1/2}}. \qquad (4.38)$$

This is $0(\Omega^{-1/2})$ for large Ω so the condition mentioned below (4.33) is satisfied by this solution. The usual form (4.37) is $0(\Omega^{-1})$. The region over which the true solution differs appreciably from (4.37) is $0(\Omega^{-1/2})$ so that $\sum_m \Delta_m$ is $0(1)$.

This procedure therefore gives an unambiguous, finite answer. From the viewpoint of perturbation theory the inclusion of the modification (4.31) to the hole propagator, without, for example, a similar modification to the particle propagator, seems rather artificial, so it is difficult to ascribe much significance to this result. Note however that for a potential which acts only in states of total momentum zero (§ 4.5), only the hole propagator is modified in this way. For such a potential, with constant matrix elements, Katz, de Shalit and Talmi (1960) have shown that this treatment gives the exact answer in the strong coupling limit. However we believe this result to be largely fortuitous, since it is not true for example when the matrix elements of v are not constant. To see this we take the strong coupling limit ($g \to \infty$) of (4.33) which gives

$$\Delta_m = g\left(v_m^2 + \sum_l v_l^2 \right). \qquad (4.39)$$

We consider a situation (see § 4.5) where the number of momentum states **k** is P, each of which can be occupied by a particle with one or two spin values. If there are N particles and if the interaction is just between particles of total momentum zero and opposite spin, then

$$\Delta E = \sum_m \Delta_m = g\left(\sum_m v_m^2 + \tfrac{1}{2} N \sum_l v_l^2 \right). \qquad (4.40)$$

This reduces to the exact answer (4.66) when v_k is independent of k, but is quite implausible in general because of the discontinuity in the dependence on v_m and v_l. Nevertheless the discussion has value in indicating with what delicacy terms of apparently negligible order in Ω^{-1} must be considered, and that the self consistent theory† may then give results which are at least finite.

4.4. *Ladder Diagrams at Finite Temperatures*

It might be suspected that the difficulties discussed above are due to concentrating attention on one of a nearly degenerate set of unperturbed states, i.e. on the unperturbed ground state which is only separated from many excited states by an energy of $0(1/\Omega)$. It is possible that perturbation theory should work better when applied to the mean energy of a band of such states. To introduce averaging over many low excited states the most obvious method is to consider the system at a small but non-zero temperature, and allow the system to become infinite before passing to the zero temperature limit. It turns out, however, that the integral equation giving the formal sum of ladder diagrams has no non-singular solution when the temperature is reduced below a certain critical value, independent of Ω for large Ω, which is, moreover, the same as the critical temperature obtained in the theory of superconductivity to be discussed in the next section.

---

† See Kümmel (1961), Wild (1960), Katz (1960 a).

We first outline briefly the general formalism for non-zero temperature, following Matsubara (1955) and Bloch and de Dominicis (1958). The grand partition function is defined by

$$Z = \mathrm{Trace}\exp[-\beta(H-\lambda N)] \quad . \quad . \quad . \quad . \quad . \quad (4.41)$$

where H is the hamiltonian operator, N is here the number operator, λ is the chemical potential and $\beta=(kT)^{-1}$, k being Boltzmann's constant and T the temperature. We write this in the form

$$Z = Z_0 \langle S(\beta) \rangle \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.42)$$

where we use the notation

$$\langle \dots \rangle = \mathrm{Tr}\,(\rho_0 \dots),$$

and where Z_0, ρ_0 are respectively the grand partition function and the statistical operator in the unperturbed system,

$$Z_0 = \mathrm{Tr}\exp[-\beta(H_0-\lambda N)] \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.43)$$

and

$$\rho_0 = Z_0^{-1}\exp[-\beta(H_0-\lambda N)]. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.44)$$

The quantity $S(\beta)$ is given by

$$S(\beta) = \exp(\beta H_0)\exp(-\beta H), \quad . \quad . \quad . \quad . \quad . \quad (4.45)$$

and satisfies

$$\frac{\partial S(\beta)}{\partial \beta} = -V(\beta)\,S(\beta) \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.46)$$

where

$$V(\beta) = \exp(\beta H_0)\;V\exp(-\beta H_0), \quad . \quad . \quad . \quad . \quad . \quad (4.47)$$

V being the perturbation.

Equation (4.46) is formally similar to the equation for the rate of change of the S matrix of field theory with time, and the methods used to simplify the perturbation expansion of the S matrix can be used here. The boundary condition is that $Z=Z_0$ at $\beta=0$. Thus

$$\frac{Z}{Z_0} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}\int_0^\beta du_1 \dots du_n \langle T[V(u_1)\dots V(u_n)]\rangle \qquad (4.48)$$

where T is the 'time ordering' operator which in this case rearranges the operators V so that they occur with decreasing values of u when read from left to right. We now use a generalization of Wick's theorem (see

Matsubara 1955, Bloch and de Dominicis 1958 and Gaudin 1960 a), which enables us to express (4.48) as the sum of contributions of all Feynmann diagrams (§ 3.3) with no external lines. We can explicitly eliminate unlinked diagrams (cf. § 3.3, see Bloch and de Dominics 1955 and Thouless 1959 b for details) and the result is conveniently expressed in terms of the thermodynamic potential $\mathscr{A}$ defined by

$$Z=\exp(-\beta\mathscr{A}). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.49)$$

We obtain

$$\mathscr{A}=\mathscr{A}_0-\frac{1}{\beta}\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\int_0^{\infty}du_1\ldots du_n\langle T[V(u_1)\ldots V(u_n)]\rangle_{\mathrm{L}} \quad (4.50)$$

where, as before, the subscript L means that only connected diagrams are to be included, and where $\mathscr{A}_0$ is the unperturbed thermodynamic potential. The contribution of a given diagram to (4.50) is obtained from the following factors:

(1) A factor $-\frac{1}{2}v(\mathbf{x}_n-\mathbf{y}_n)$ for an interaction line from $\mathbf{x}_n$ to $\mathbf{y}_n$.

(2) A factor $G^0(\beta;\, x,\, x')$ for a line from an upper end $x'(\equiv\mathbf{x}',\, u')$ to an upper end x, and similar factors $G^0(\beta;\, y,\, y')$ etc. for a line from a lower end y' to a lower end y, etc.

(3) A factor (-1) for each closed loop.

These factors must be integrated over all space and over values of u from 0 to β, and multiplied by $(\beta n!)^{-1}$. The propagator G^0 is given, in momentum space, by

$$\begin{aligned} G_k{}^0(\beta;u,u') &= -\frac{\exp((k^2/2M)-\lambda)(u-u')}{1+\exp\beta(k^2/2M-\lambda)}, \quad u'>u, \\ &= \frac{\exp((k^2/2M)-\lambda)(u-u')}{1+\exp-\beta(k^2/2M-\lambda)}, \quad u'<u. \qquad (4.51) \end{aligned}$$

In the manner of § 3.7 it is possible to introduce an effective field modification to the propagators. We will not go into details but will allow for this possibility by replacing $k^2/2M$ by e_k. We shall use the notation

$$\epsilon(k)=\epsilon_k=e_k-\lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.52)$$

Following Thouless (1960 b) we consider the sum of all generalized ladders, i.e. diagrams in which two lines enter and pass through all vertices in the diagram. A typical diagram of this kind is shown below (3.97). We denote by $L_{\mathbf{n}\,\mathbf{m}}(\mathbf{K};\, u',\, u)$ the sum of all such diagrams where the incoming lines have momenta $\mathbf{K}\pm\mathbf{m}$ and the emerging lines momenta $\mathbf{K}\pm\mathbf{n}$. Then, considering the case where the particles have opposite spin, we have

$$\begin{aligned} L_{\mathbf{n}\,\mathbf{m}}(\mathbf{K};u',u) &= G_{\mathbf{K}+\mathbf{n}}{}^0(\beta;u',u)G_{\mathbf{K}-\mathbf{n}}{}^0(\beta;u',u)\,\delta_{\mathbf{n}\,\mathbf{m}} \\ &\quad -\sum_{\mathbf{k}}\int_0^{\beta}du\,G_{\mathbf{K}+\mathbf{n}}{}^0(\beta;u',u'')G_{\mathbf{K}-\mathbf{n}}{}^0(\beta;u',u'')\langle\mathbf{n}|v|\mathbf{k}\rangle \\ &\quad \times L_{\mathbf{k}\,\mathbf{m}}(\mathbf{K};u'',u) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.53) \end{aligned}$$

It follows (Thouless 1960) from this equation that L depends on u, and u' only through $(u-u')$ and that it is periodic in $u-u'$ with period β. Thus we can write

$$L_{\mathbf{n}\,\mathbf{m}}(\mathbf{K}\,;u,u') = \sum_{\gamma=-\infty}^{\infty} L_{\mathbf{n}\,\mathbf{m}}(\mathbf{K},\nu)\exp\left[2\pi\nu i(u-u')/\beta\right] \qquad (4.54)$$

where

$$L_{\mathbf{n}\,\mathbf{m}}(\mathbf{K},v) = \frac{\tanh\left[\frac{1}{2}\beta\epsilon(\mathbf{k}+\mathbf{n})\right]+\tanh\left[\frac{1}{2}\beta\epsilon(\mathbf{k}-\mathbf{n})\right]}{2\beta\left[\epsilon(\mathbf{k}+\mathbf{n})+\epsilon(\mathbf{k}-\mathbf{n})\right]-4\pi i\nu}$$
$$\times\left[\delta_{\mathbf{n}\,\mathbf{m}}-\beta\sum_{\mathbf{k}}\langle\mathbf{n}\,|\,v\,|\,\mathbf{k}\rangle L_{\mathbf{k}\,\mathbf{m}}(\mathbf{K},\nu)\right]. \quad . \quad . \quad . \quad . \qquad (4.55)$$

This equation has a non-singular solution only above a certain critical temperature. Assuming the singularity to appear first for $\mathbf{k}=v=0$, this temperature is the lowest temperature above which the homogeneous equation

$$\chi_{\mathbf{k}} = -\frac{\tanh\left(\frac{1}{2}\beta\epsilon_{\mathbf{k}}\right)}{2\epsilon_{\mathbf{k}}}\sum_{\mathbf{k}}\langle\mathbf{k}\,|\,v\,|\,\mathbf{k}'\rangle\,\chi_{\mathbf{k}'} \quad . \quad . \quad . \quad . \qquad (4.56)$$

has no non-trivial solution. This critical temperature is identical with that of the Bardeen–Cooper–Schrieffer superconductivity theory (§ 4.6).

Thouless shows that there is always a non-zero critical temperature for an attractive separable potential. More general potentials are considered by Emery (1960). He finds in particular that there is a non-zero critical temperature when the Brueckner K matrix is singular; the converse of this does not hold.

The statistical perturbation theory reduces in a trivial way to the previous ground state formalism when the temperature is made zero for fixed Ω. To see this it is essential to note that with finite Ω at zero temperature the particle number is a discontinuous function of λ and the interaction strength. The more interesting case, that the limit $\Omega\to\infty$ is taken before $\beta\to\infty$, requires more discussion (Kohn and Luttinger 1960, Luttinger and Ward 1960, Bloch 1960 b).

4.5. *The Bardeen–Cooper–Schrieffer Theory*

The divergence of the K matrix arises because, for the interaction of particles with momenta sufficiently near to the fermi surface, the allowed energy denominators can be indefinitely small compared to the strength of the interaction. Such a situation corresponds to a 'strong coupling' problem, and suggests that further understanding might be obtained by consideration of the strong coupling limit. To this end we first consider a system with $2P$ degenerate single particle states $|\,\mathbf{k}\,\rangle$; $\mathbf{k}=\pm\mathbf{k}_1,\ \pm\mathbf{k}_2,\ldots\pm\mathbf{k}_P$, where the state $+\mathbf{k}$ has opposite spin and momentum to $-\mathbf{k}$. We ignore isobaric spin in this example. Let the interaction be between pairs of particles with total momentum and total spin zero, and be otherwise independent of $\mathbf{k}$, i.e.

$$V = g\sum_{\mathbf{k},\mathbf{k}'} a_{\mathbf{k}}^{+}a_{-\mathbf{k}}^{+}a_{-\mathbf{k}'}a_{\mathbf{k}'} \quad . \quad . \quad . \quad . \quad . \qquad (4.58)$$

where the sums are over $\mathbf{k}$ and $\mathbf{k}'$ equal to $\mathbf{k}_1, \mathbf{k}_2 \ldots \mathbf{k}_P$. We put

$$b_{\mathbf{k}} = a_{-\mathbf{k}} a_{\mathbf{k}}, \quad b_{\mathbf{k}}^{+} = a_{\mathbf{k}}^{+} a_{-\mathbf{k}}^{+}; \quad \ldots \ldots \quad (4.59)$$

then

$$V = g \sum_{\mathbf{k}, \mathbf{k}'} b_{\mathbf{k}}^{+} b_{\mathbf{k}'}. \quad \ldots \ldots \quad (4.60)$$

It is easy to see that the pairs of particles corresponding to the b operators obey Bose statistics when the momenta are different, i.e.

$$[b_{\mathbf{k}}, b_{\mathbf{k}'}^{+}]_{-} = [b_{\mathbf{k}}, b_{\mathbf{k}'}] = [b_{\mathbf{k}}^{+}, b_{\mathbf{k}'}^{+}] = 0, \quad \text{for} \quad \mathbf{k} \neq \mathbf{k}'. \quad \ldots \quad (4.61)$$

For $\mathbf{k} = \mathbf{k}'$ they obey fermi statistics provided the space of states is limited to that for which only pair occupation is allowed (i.e. if the state $+k$ is occupied then $-k$ is also occupied and vice versa). Thus, in this restricted space,

$$[b_{\mathbf{k}}, b_{\mathbf{k}}^{+}]_{+} = 1. \quad \ldots \ldots \quad (4.62)$$

Since the hamiltonian can be expressed entirely in terms of the b operators, the energy eigenstates are in the restricted space in which only pair occupation is allowed. As a simple example suppose there are only two particles in the system, then the basis states are $b_{\mathbf{k}}^{+} | \text{vac.} \rangle$ and, if we take their unperturbed energy (equal for all $\mathbf{k}$ by hypothesis) to be zero, we have for the matrix elements of the hamiltonian

$$\langle \mathbf{k}' | H | \mathbf{k} \rangle = g. \quad \ldots \ldots \quad (4.63)$$

This can easily be diagonalized; for an attractive potential all the eigenvalues are zero except the lowest which is $-Pg$. The ground state is given by

$$\left(\sum_{\mathbf{k}} b_{\mathbf{k}}^{+} \right) | \text{vac.} \rangle.$$

By analogy we expect the ground state for N particles to be given by

$$| \mathbf{0} \rangle = \left(\sum_{\mathbf{k}} b_{\mathbf{k}}^{+} \right)^{N/2} | \text{vac.} \rangle. \quad \ldots \ldots \quad (4.64)$$

Operating on this with the hamiltonian we obtain

$$H | \mathbf{0} \rangle = g \frac{N}{2} \left(P - \frac{N}{2} + 1 \right) | \mathbf{0} \rangle; \quad \ldots \ldots \quad (4.65)$$

so that it is indeed an eigenstate, with energy

$$\Delta E = g \frac{N}{2} \left(P - \frac{N}{2} + 1 \right). \quad \ldots \ldots \quad (4.66)$$

For a proof that it is the lowest eigenstate for $g < 0$, and for a derivation of the complete spectrum we refer to the original papers (Wada *et al.* 1958, Mottleson 1958, Thouless 1960 a, b and Katz *et al.* 1960).

We now turn to the general problem where the single particle states are not degenerate, but we still restrict the interaction to be between pairs with equal and opposite momenta. With nucleons we can consider parallel or antiparallel pairings of isobaric as well as ordinary spin. For simplicity we take both antiparallel. Indicating ordinary spin explicitly and absorbing isospin into the symbol k we have then the 'reduced hamiltonian'

$$H^{\mathrm{BCS}} = \sum_{\mathbf{k}} e_{\mathbf{k}}(a_{\mathbf{k}\uparrow}{}^{+}a_{\mathbf{k}\uparrow} + a_{\mathbf{k}\downarrow}{}^{+}a_{\mathbf{k}\downarrow}) + \sum_{\mathbf{k}\,\mathbf{k}'} (\mathbf{k}\,|\,v\,|\,\mathbf{k}')\, a_{\mathbf{k}\uparrow}{}^{+}a_{-\mathbf{k}\downarrow}{}^{+}a_{-\mathbf{k}'\downarrow}a_{\mathbf{k}'\uparrow}. \quad . \quad . \quad . \quad (4.67)$$

The e_k are single particle energies which may contain effective field modifications from parts of the interaction not now considered explicitly.

The interaction retained in (4.67) is of order Ω^{-1} relative to the total interaction. It is found however to cause an energy shift per particle independent of Ω. It is hoped that certain qualitative features of the eigenstates of H^{BCS} may remain true for the total hamiltonian. One cannot be entirely confident of this because the neglect of interaction in states with total momentum not zero gives the hamiltonian a peculiar character—in configuration space it has an infinite range of non-locality. Incidentally it should be noted that although H^{BCS} allows the occupant of a given state to interact with that of only one other state, in perturbation theory it generates other diagrams in addition to the ladders.

The natural generalization of (4.64) to take account of the different energies of the unperturbed states would be

$$\left(\sum_{\mathbf{k}} w_{\mathbf{k}} a_{\mathbf{k}\uparrow}{}^{+}a_{-\mathbf{k}\downarrow}{}^{+}\right)^{N/2} |\,\mathrm{vacuum}\,\rangle$$

with $w_{\mathbf{k}}$ to be determined variationally. However it is more convenient to work with the state (Bardeen *et al.* 1957, referred to as BCS):

$$|\,\Psi^{\mathrm{BCS}}\rangle = \prod_{\mathbf{k}} \frac{[1+g_{\mathbf{k}}\, a_{\mathbf{k}\uparrow}{}^{+}a_{-\mathbf{k}\uparrow}{}^{+}]}{[1+|\,g_{\mathbf{k}}\,|^2]^{1/2}}\, |\,\mathrm{vacuum}\,\rangle \quad . \quad . \quad . \quad (4.68)$$

where the product is over all momenta and isobaric spins. The variation parameters g are related to the occupation probabilities h for individual states by

$$h_{\mathbf{k}} = |\,g_{\mathbf{k}}\,|^2(1+|\,g_{\mathbf{k}}\,|^2)^{-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.69)$$

The state (4.68) is not an eigenstate of particle number, but turns out to be sharply peaked about the mean value; the projection on to a state of definite particle number has the required form.

We must minimize the energy expectation value for given mean number of particles. Introducing the Lagrange multiplier λ we find

$$\begin{aligned}\langle\Psi^{\mathrm{BCS}}|\,H^{\mathrm{BCS}}-\lambda N\,|\Psi^{\mathrm{BCS}}\rangle \\ = \sum_{\mathbf{k}} 2\epsilon_{\mathbf{k}} \frac{|\,g_{\mathbf{k}}\,|^2}{1+|\,g_{\mathbf{k}}\,|^2} + \sum_{\mathbf{k}\,\mathbf{k}'} \langle\mathbf{k}\,|\,v\,|\,\mathbf{k}'\rangle \frac{g_{\mathbf{k}}{}^{*}}{(1+|\,g_{\mathbf{k}}\,|^2)}\frac{g_{\mathbf{k}'}}{(1+|\,g_{\mathbf{k}'}\,|^2)} \\ + \sum_{\mathbf{k}} \langle\mathbf{k}\,|\,v\,|\,\mathbf{k}\rangle \frac{|\,g_{\mathbf{k}}\,|^4}{(1+|\,g_{\mathbf{k}}\,|^2)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.70)\end{aligned}$$

where as before

$$\epsilon_{\mathbf{k}} = e_{\mathbf{k}} - \lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.71)$$

The last term of (4.70) is of order Ω^{-1} relative to the others and will be ignored. Variation of the $g_{\mathbf{k}}$ yields the stationary conditions

$$\mu_{\mathbf{k}}^* g_{\mathbf{k}}^2 + 2\epsilon_k g_k - \mu_{\mathbf{k}} = 0 \qquad (4.72)$$

where

$$\mu_{\mathbf{k}} = -\sum_{\mathbf{k}'} \langle \mathbf{k} | v | \mathbf{k}' \rangle \frac{g_{\mathbf{k}'}}{1+|g_{\mathbf{k}'}|^2} \qquad (4.73)$$

and where we have assumed $\langle \mathbf{k} | v | \mathbf{k}' \rangle$ to be hermitian. The solution of (4.72) is

$$g_{\mathbf{k}} = \frac{1}{\mu_{\mathbf{k}}^*} [(\epsilon_{\mathbf{k}}^2 + |\mu_{\mathbf{k}}|^2)^{1/2} - \epsilon_{\mathbf{k}}] \qquad (4.74)$$

where we have taken the positive square root since, as we see by forming the second derivative of (4.70), the other root gives instability. Then, from (4.69),

$$h_{\mathbf{k}} = \tfrac{1}{2}\left[1 - \frac{\epsilon_{\mathbf{k}}}{[\epsilon_{\mathbf{k}}^2 + |\mu_{\mathbf{k}}|^2]^{1/2}}\right]. \qquad (4.75)$$

This is approximately unity for $\epsilon_k \ll -\mu_{\mathbf{k}}$ and zero for $\epsilon_{\mathbf{k}} \gg \mu_{\mathbf{k}}$, and varies smoothly between these limits, so that the sharp cut-off at the fermi surface for the unperturbed gas is smoothed out by the interaction (provided $\mu_{\mathbf{k}} \neq 0$ of course).

If we put (4.74) into (4.73) we obtain

$$\mu_{\mathbf{k}} = -\sum_{\mathbf{k}'} \frac{\langle \mathbf{k} | v | \mathbf{k}' \rangle \mu_{\mathbf{k}'}}{2[\epsilon_{k'}^2 + |\mu_{\mathbf{k}'}|^2]^{1/2}}. \qquad (4.76)$$

One solution of this is the trivial one $\mu_{\mathbf{k}} = 0$, for which we obtain the energy and occupation probability of the unperturbed state. In certain circumstances this equation also has a non-trivial solution. If this solution leads to a lower ground-state energy than the unperturbed energy, it follows that the true ground-state energy, being at least as low as this, is lower than the unperturbed energy and therefore unattainable by perturbation theory (which gives a zero energy shift for H^{BCS}).

As a simple example which shows the above phenomena, BCS take a spin independent potential of the form

$$\begin{aligned} \langle \mathbf{k}' | v | \mathbf{k} \rangle &= \frac{-v_0}{\Omega} \text{ for } \quad k_f - t < k < k_f + t, \\ &\qquad \text{and } k_f - t < k' < k_f + t, \qquad (4.77) \\ &= 0 \text{ otherwise,} \end{aligned}$$

where $t \ll k_f$. Then, from (4.73), $\mu_{\mathbf{k}} = \epsilon_0$, say, is independent of $\mathbf{k}$ and (4.76) becomes

$$\epsilon_0 = \tfrac{1}{2} \frac{v_0}{\Omega} \epsilon_0 \sum_{\mathbf{k}}{}' \frac{1}{(\epsilon_k^2 + \epsilon_0^2)^{1/2}} \qquad (4.78)$$

where $\sum'$ means the sum over all momentum states with $k_f - t < k < k_f + t$.

Obviously this has a non-trivial solution only if the potential is attractive, $v_0 > 0$, when, in the limit of infinite volume,

$$\epsilon_0 = \frac{\delta e}{\sinh(1/n_f v_0)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4.79)$$

where Ωn_f is the density of momentum states per unit energy interval near the fermi surface, given by

$$\Omega n_f = \left[\frac{\partial e(k)}{\partial k}\bigg|_{k_f}\right]^{-1} \frac{3A}{k_f} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4.80)$$

and

$$\delta e = |e(k_f + t) - e(k_f)| = |e(k_f) - e(k_f - t)|. \quad . \quad . \quad . \qquad (4.81)$$

To find the energy it is necessary to calculate λ. Alternatively we can put $\lambda = e_f$ (its value in the unperturbed state) and then calculate the number of particles being considered. With $\lambda = e_f$ there follows from (4.70), (4.75) and (4.79) after some calculation

$$\langle \Psi^{\mathrm{BCS}} | H^{\mathrm{BCS}} - \lambda N | \Psi^{\mathrm{BCS}} \rangle - \sum_{\mathbf{m}} (e_m - \lambda)$$
$$= \frac{-2n_f(\delta e)^2}{\exp[2/n_f v_0] - 1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4.82)$$

We first use this result to show that the system must have an exact ground-state energy which is lower then the unperturbed value, and hence unattainable by perturbation theory. To see this we note that, for any normalised Ψ,

$$\langle \Psi | H^{\mathrm{BCS}} - \lambda N | \Psi \rangle \geqslant E(\bar{N}) - \lambda \bar{N} \quad . \quad . \quad . \quad . \quad . \qquad (4.83)$$

where $E(\bar{N})$ is the exact ground state energy of H^{BCS} for $\bar{N}$ particles, and $\bar{N}$ is the solution of

$$\lambda = e_f = \frac{\partial E(\bar{N})}{\partial \bar{N}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4.84)$$

If the unperturbed energy is equal to the exact energy then the solution of (4.84) is the number of particles in the unperturbed state, i.e.

$$\bar{N} = \sum_{\mathbf{m}} 1 = N_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4.85)$$

In this case (4.82) contradicts (4.83) with Ψ^{BSC} for Ψ. Since the unperturbed energy cannot be lower than the exact energy the result follows.

In general (4.85) is not satisfied so that (4.82) does not immediately give the variational approximation to the energy shift. It does, however, give it correctly to first order in the energy shift, since, to this order,

$$E(\bar{N}) \simeq E(N_0) + \lambda(\bar{N} - N_0).$$

Then, from (4.83),

$$\langle \Psi^{\mathrm{BCS}} | H^{\mathrm{BCS}} - \lambda N | \Psi^{\mathrm{BCS}} \rangle - \sum_m (e_m - \lambda) \geqslant E(N_0) - \sum_m e_m = \Delta E(N_0).$$
$$. \quad . \quad . \qquad (4.86)$$

When the BCS trial wavefunction is used for the strong coupling problem discussed at the beginning of this section (Katz *et al.* 1960), $\mu_{\mathbf{k}}$ and $g_{\mathbf{k}}$ given by (4.74) and (4.76) become independent of **k**, and are thus determined by the condition that

$$\sum_{\mathbf{k}} h_{\mathbf{k}} = 2P \frac{|g_{\mathbf{k}}|^2}{1+|g_{\mathbf{k}}|^2} = N. \qquad (4.87)$$

Substituting for $g_{\mathbf{k}}$ in (4.68) yields on evaluation of the energy

$$\Delta E = g\frac{N}{2}\left(P - \frac{N}{2} + \frac{N}{2P}\right) \qquad (4.88)$$

which is to be compared with the exact answer in (4.66). The difference is due to the fact that the BCS trial state gives an average over a small region of particle numbers. The error becomes negligible as $P \to \infty$ (assuming $P - N/2$ is not $0(1)$).

In § 4.8 we shall investigate whether (4.76) has a non-trivial solution for nuclear forces. Here we shall assume such a solution exists and consider the possible properties of the system. To do this it is convenient to introduce the idea of 'quasi-particles' by means of the Bogoliubov–Valatin transformation (Bogoliubov 1947, 1958 a, b and Valatin 1958).

We put

$$\left.\begin{aligned} \xi_{\mathbf{k}\uparrow} &= \frac{a_{\mathbf{k}\uparrow} - g_{\mathbf{k}} a_{-\mathbf{k}\downarrow}{}^{+}}{[1+|g_{\mathbf{k}}|^2]^{1/2}}, \\ \xi_{\mathbf{k}\downarrow} &= \frac{a_{-\mathbf{k}\downarrow} + g_{\mathbf{k}} a_{\mathbf{k}\uparrow}{}^{+}}{[1+|g_{\mathbf{k}}|^2]^{1/2}}. \end{aligned}\right\} \qquad (4.89)$$

It is easily seen that for any $g_{\mathbf{k}}$ the operators $\xi_{\mathbf{k}}$ and $\xi_{\mathbf{k}\downarrow}$ obey the fermion commutation relations and that (4.89) is a canonical transformation. The inverse transformation is given by

$$\left.\begin{aligned} a_{\mathbf{k}\uparrow} &= \frac{\xi_{\mathbf{k}\uparrow} + g_{\mathbf{k}} \xi_{\mathbf{k}\downarrow}{}^{+}}{[1+|g_{\mathbf{k}}|^2]^{1/2}}, \\ a_{\mathbf{k}\downarrow} &= \frac{\xi_{-\mathbf{k}\downarrow} - g_{\mathbf{k}} \xi_{-\mathbf{k}\uparrow}{}^{+}}{[1+|g_{\mathbf{k}}|^2]^{1/2}}. \end{aligned}\right\} \qquad (4.90)$$

If the $g_{\mathbf{k}}$ are identified with those in (4.68) the state $|\Psi^{\mathrm{BCS}}\rangle$ is the 'quasi-particle vacuum', i.e.

$$\xi_{\mathbf{k}\sigma}|\Psi^{\mathrm{BCS}}\rangle = 0. \qquad (4.91)$$

In the BCS theory the quasi-particles are regarded as independent particles, in which case the excited states of the system are given by

$$\xi_{\mathbf{k}_1\sigma_1}{}^{+}\xi_{\mathbf{k}_2\sigma_2}{}^{+}\dots\xi_{\mathbf{k}_n\sigma_n}{}^{+}|\Psi^{\mathrm{BCS}}\rangle, \quad n = 1, 2, \dots \qquad (4.92)$$

Various attempts (see below) have been made to justify the use of these states as the excited states—they are not of course exact eigenstates of H^{BCS}; the real justification, however, lies in the success the theory has had in providing a basis for the theory of superconductors. To find the 'energy' of the states (4.92), i.e. the expectation value of the hamiltonian in the states, we insert the transformation (4.89) into (4.67). Then, ignoring the interaction between quasi-particle, which we can do if n is small, i.e. $0(1)$ rather than $0(\Omega)$ for large Ω, the part of the hamiltonian which is diagonal in the space of states (4.92) is

$$\begin{aligned}(H^{\mathrm{BCS}}-\lambda N)_{\mathrm{diag.}} &= \sum_{\mathbf{k}} \epsilon_k \frac{|g_{\mathbf{k}}|^2}{1+|g_{\mathbf{k}}|^2} + \tfrac{1}{2} \sum_{\mathbf{k},\,\mathbf{k}'} \frac{g_{\mathbf{k}'} g_{\mathbf{k}}^* \langle \mathbf{k} | v | \mathbf{k}' \rangle}{[1+|g_{\mathbf{k}'}|^2][1+|g_{\mathbf{k}}|^2]} \\ &+ \tfrac{1}{2} \sum_{\mathbf{k}} \left[\frac{\epsilon_k (1-|g_{\mathbf{k}}|^2)}{1+|g_{\mathbf{k}}|^2} - \sum_{\mathbf{k}'} \frac{(g_{\mathbf{k}}^* g_{\mathbf{k}'} + g_{\mathbf{k}} g_{\mathbf{k}'}^*) \langle \mathbf{k} | v | \mathbf{k}' \rangle}{[1+|g_{\mathbf{k}'}|^2][1+|g_{\mathbf{k}}|^2]} \right] \\ &\quad (\xi_{\mathbf{k}\uparrow}^{+} \xi_{\mathbf{k}\uparrow} + \xi_{\mathbf{k}\downarrow}^{+} \xi_{\mathbf{k}\downarrow}). \qquad \text{(4.93)}\end{aligned}$$

Using (4.73) and (4.74) we find for the energy of the states (4.92)

$$E(\{\mathbf{k}_i\}) = \sum_i [\epsilon_{k_i}{}^2 + |\mu_{\mathbf{k}_i}|^2]^{1/2} + E^{\mathrm{BCS}} \qquad \text{(4.94)}$$

where E^{BCS} is the ground-state energy in the BCS approximation.

The lowest excitation energy will be obtained for a state with one quasi-particle with (for a reasonable potential) $|\mathbf{k}| = k_f$. Its excitation energy will be $\mu_{\mathbf{k}_f}$. We can expand $\mu_{\mathbf{k}}$ in partial waves (Brueckner, Soda, Anderson and Morel 1960, Anderson and Morel 1960)

$$\mu_{\mathbf{k}} = \mu_k{}^{l,\,m} \sum_{l,\,m} Y_l{}^m (\hat{\mathbf{k}}, \hat{\mathbf{k}}_0) \qquad \text{(4.95)}$$

where $\mathbf{k}_0$ is some fixed vector, and suppose (see § 4.7) that the non-trivial solution occurs for just one partial wave. The particular $\mu_{\mathbf{k}}{}^{l,\,m}$ will then depend just on the interaction in the lth partial wave state. If this is an s-state it will lead to an energy gap in the spectrum of the complete system since the lowest energy will then be that of a particle–hole pair with energy

$$2\epsilon_0 = 2\mu_{k_f}{}^{00}. \qquad \text{(4.96)}$$

For other values of l the energy spectrum of single quasi-particles will be given, in this approximation, by

$$\mu_{\mathbf{k}} = \mu_k{}^{l,\,m}\, Y_l{}^m (\hat{\mathbf{k}}, \hat{\mathbf{k}}_0). \qquad \text{(4.97)}$$

For suitable directions this will go to zero so there will not be an energy gap but only a reduction in the density of low lying states. For these cases the material will become anisotropic. Some discussion is given by Anderson and Morel (1960).

There are alternative ways of obtaining the results of this section. Bogoluibov (1958 a) used the transformation (4.89) and chose the $g_{\mathbf{k}}$ so that no zero energy denominators arose in a perturbation expansion starting from $|\Psi^{\mathrm{BCS}}\rangle$ as the unperturbed state. Bayman (1959) used a trial wavefunction with a fixed number of particles and evaluated the expectation value of H^{BCS} by a saddle point approximation.

4.6. *BCS Theory at Finite Temperatures*

We briefly outline the generalization of the theory to finite temperature, following BCS and Valatin (1958). It can be shown (Peierls 1938, Valatin 1958) that it is necessary in this case to minimize the thermodynamic potential

$$\mathscr{A}=\overline{\langle H^{\mathrm{BCS}}-\lambda N\rangle}-TS \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.98)$$

where the bar denotes an ensemble average and S is the entropy. We calculate these quantities by assuming we have a fermi gas of quasi-particles, defined by (4.89). Then

$$TS=-\beta\sum_{\mathbf{k}}\{f_{\mathbf{k}}\log f_{\mathbf{k}}+(1-f_{\mathbf{k}})\log(1-f_{\mathbf{k}})\} \quad . \quad . \quad . \quad (4.99)$$

where

$$f_{\mathbf{k}}=\overline{\langle\xi_{\mathbf{k}\uparrow}{}^{+}\xi_{\mathbf{k}\uparrow}\rangle}=\overline{\langle\xi_{\mathbf{k}\downarrow}{}^{+}\xi_{\mathbf{k}\downarrow}\rangle}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.100)$$

Using (4.67) and (4.89) we obtain

$$\mathscr{A}=\sum_{\mathbf{k}}\epsilon_k f_{\mathbf{k}}+\sum_{\mathbf{k}}\epsilon_k\frac{|g_{\mathbf{k}}|^2}{1+|g_{\mathbf{k}}|^2}(1-2f_{\mathbf{k}})$$

$$+\tfrac{1}{2}\sum_{\mathbf{k},\mathbf{k}'}\langle\mathbf{k}|v|\mathbf{k}'\rangle\frac{g_{\mathbf{k}}{}^{*}}{1+|g_k|^2}\cdot\frac{g_{\mathbf{k}'}}{1+|g_{k'}|^2}(1-2f_{\mathbf{k}})(1-2f_{\mathbf{k}'})-TS.$$

$$. \quad . \quad . \quad (4.101)$$

On minimizing with respect to $g_{\mathbf{k}}$ as before we obtain (4.74) but with (4.76) replaced by

$$\mu_{\mathbf{k}}=-\sum_{\mathbf{k}'}\frac{\langle\mathbf{k}|v|\mathbf{k}'\rangle(1-2f_{\mathbf{k}'})}{2[\epsilon_{k'}{}^2+|\mu_{\mathbf{k}'}|^2]^{1/2}}\mu_{\mathbf{k}'}. \quad . \quad . \quad . \quad . \quad (4.102)$$

Minimization with respect to $f_{\mathbf{k}}$ yields

$$f_{\mathbf{k}}=[1+\exp\beta\{\epsilon_k{}^2+|\mu_{\mathbf{k}}|^2\}^{1/2}]^{-1}, \quad . \quad . \quad . \quad . \quad . \quad (4.103)$$

whence (4.102) becomes

$$\mu_{\mathbf{k}}=-\sum_{\mathbf{k}'}\frac{\langle\mathbf{k}|v|\mathbf{k}'\rangle}{2[\epsilon_{k'}{}^2+|\mu_{\mathbf{k}'}|^2]^{1/2}}\tanh[\tfrac{1}{2}\beta(\epsilon_{k'}{}^2+|\mu_{\mathbf{k}'}|^2)^{1/2}]\,\mu_{\mathbf{k}'}. \qquad (4.104)$$

This can be written in a more convenient form by introducing the definition

$$\chi_{\mathbf{k}}=\frac{\tanh[\tfrac{1}{2}\beta(\epsilon_k{}^2+|\mu_{\mathbf{k}}|^2)^{1/2}]}{2[\epsilon_k{}^2+|\mu_{\mathbf{k}}|^2]^{1/2}}\mu_{\mathbf{k}} \quad . \quad . \quad . \quad . \quad (4.105)$$

giving

$$\chi_{\mathbf{k}}=-\tfrac{1}{2}\frac{\tanh[\tfrac{1}{2}\beta(\epsilon_k{}^2+|\mu_{\mathbf{k}}|^2)^{1/2}]}{[\epsilon_k{}^2+|\mu_{\mathbf{k}}|^2]^{1/2}}\sum_{\mathbf{k}'}\langle\mathbf{k}|v|\mathbf{k}'\rangle\chi_{\mathbf{k}'}. \qquad (4.106)$$

The critical temperature $T_c=1/k\beta_c$ is the highest temperature below which this equation has a non-trivial solution; above this temperature the thermodynamics is that of an ideal fermi gas. We can obtain T_c more simply from the linear eigenvalue equation

$$\chi_{\mathbf{k}}=-\mu\frac{\tanh(\tfrac{1}{2}\beta\epsilon_k)}{2|\epsilon_k|}\sum_{\mathbf{k}'}\langle\mathbf{k}|v|\mathbf{k}'\rangle\chi_{\mathbf{k}'}. \quad . \quad . \quad . \quad (4.107)$$

By arguments like those of § 4.8 the existence of an eigenvalue in the range $0<\mu<1$ is equivalent to the existence of a non-trivial solution of (4.106). For all eigenvalues (Emery 1960) $|\mu|$ increases monotonically to infinity with increase of temperature. Thus T_c is the highest temperature at which (4.107) has an eigenvalue $\mu=1$. Below this temperature there is always a non-trivial solution of (4.106), and nowhere above it. Comparing (4.107) with (4.56), T_c is also the temperature below which the sum of generalized ladders is divergent. It follows (Emery 1960) that when the Brueckner K matrix is singular the BCS theory gives a supernormal state at zero temperature.

Bogoliubov *et al.* (1958) have obtained these results by dividing H^{BCS} into two parts, one of which is exactly soluble. For two different choices of the soluble part the effect of the remainder is of relative order Ω^{-1} in every order of perturbation theory. These give respectively the unperturbed and BCS free energies, and it was asserted that both solutions are exact for a large system. But such reasoning is clearly not conclusive (see Wentzel 1960). On rigorous examination (Bogoliubov 1960) the unperturbed solution has been rejected, and the asymptotic exactness of the BCS solution reaffirmed.

Gaudin (1960 b) has found a part of the diagram expansion in statistical perturbation theory which yields the BCS result on formal summation. It has also been obtained by Martin and Kadanoff (1961), following Gorkov (1958), with the Green function method.

4.7. *The Effect of the Residual Terms in the Hamiltonian*

Here we consider briefly the effects of $H-H^{\mathrm{BCS}}$. Although this is the main part of the potential it is hoped—and this is one of the principal assumptions of the BCS theory of superconductivity–that its inclusion will not qualitatively upset the main features of the above model.

First of all it is necessary to decide which particular spin and isobaric spin state should be considered in the interaction retained in H^{BCS}. It seems reasonable to include that corresponding to the 'strongest' attraction, where this might be defined as that which gives the largest critical temperature.

Thouless (1960 b) makes the quasi-particle transformation (4.89) and then considers $H-H^{\mathrm{BCS}}$ as a perturbation on the quasi-particle vacuum. He finds that the energy gap and smoothing out of the fermi surface associated with the previous solutions of H^{BCS} are sufficient to make the sum of quasi-particle ladder diagrams, with total momentum different from zero and spin equal to that of the interaction retained in H^{BCS}, finite at all temperatures. In fact this sum goes as $\log p$, where p is the total momentum, for small $p(\neq 0)$. For $p=0$ the sum is divergent as previously stated. The sum of ladder diagram with different total spins is finite at the critical temperature but not necessarily at all lower temperatures. Thouless (1960 b) suggests that if it becomes divergent it probably implies the formation of 'bound clusters' of more than two particles. We know of no investigation of this problem appropriate to nuclear matter.

Anderson (1958) has shown that the longitudinal oscillations, generated by the operator

$$\rho_{\mathbf{q}} = \sum_{\mathbf{k}} a_{\mathbf{k}+\mathbf{q}}{}^{+} a_{\mathbf{k}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4.109)$$

acting on $|\Psi^{\mathrm{BCS}}\rangle$, have energy expectation values (of H) continuously distributed above the BCS ground state energy. Thus the energy gap is removed by these collective excitations, although the density of states remains much reduced. For metals the coulomb interaction has the effect of raising the energy of these collective state so that they do not fill the gap. The collective states are important in considerations of the electromagnetic properties of superconductors. For these and other details relevant to superconductors we refer to the original papers mentioned above and also to Kuper (1959), Abrikosov and Khalatnikov† (1959), Bogoliubov *et al.* (1959), Rickayzen (1959), Nambu (1960) and Wentzel (1960) where further references are given.

As we have stated the main justification for the BCS model lies in its success in treating superconductivity in metals and therefore, since the physical situation in nuclei is not the same as that in metals, we may not be justified in applying the model to nuclear matter. In particular we have to remember that the hard core demands considerable further correlation beyond that contained in Ψ^{BCS}. Further it is known that self-consistency effects are important in nuclei and the BCS theory gives no prescription for dealing with such effects. In view of these points, and the fact (see §§ 4.2, 4.8) that the existence or not of a non-trivial solution of the BCS integral equation depends sensitively upon the treatment of self-consistency, it seems premature to make confident assertions that nuclear matter is, or is not, a superfluid.

4.8. *The Superfluidity of Nuclear Matter*

With the above qualifications, we now discuss whether the forces in nuclear matter are such as to cause superfluidity‡. To this end we could consider solutions of (4.76); alternatively we can argue as follows. We introduce

$$\chi_{\mathbf{k}} = \frac{g_{\mathbf{k}}}{1+|g_{\mathbf{k}}|^2} = \frac{\mu_{\mathbf{k}}}{2[\epsilon_k{}^2+|\mu_{\mathbf{k}}|^2]^{1/2}} \quad . \quad . \quad . \quad . \qquad (4.110)$$

then, using (4.70),

$$\langle\Psi^{\mathrm{BCS}}|H^{\mathrm{BCS}}-\lambda N|\Psi^{\mathrm{BCS}}\rangle - \sum_m (e_m-\lambda) = \sum_{\mathbf{k}} |\epsilon_{\mathbf{k}}|\{\tfrac{1}{2}-\tfrac{1}{2}(1-4|\chi_{\mathbf{k}}|^2)^{1/2}\}$$
$$+\tfrac{1}{2}\sum_{\mathbf{k},\mathbf{k}'}\langle\mathbf{k}|v|\mathbf{k}'<\chi_{\mathbf{k}}{}^{*}\chi_{\mathbf{k}'}. \qquad (4.111)$$

Then superfluidity occurs whenever there exists a χ which makes (4.111) less than zero. A necessary and sufficient condition for this is that there should exist a $\chi_{\mathbf{k}}$ such that

---

† Abrikosov, A. A., and Khalatnikov, I. M., *Advanc. Phys.*, **8**, 45.

‡ See also de Dominicis and Martin 1958.

$$\sum_{\mathbf{k}} |\chi_{\mathbf{k}}|^2 |\epsilon_k| + \tfrac{1}{2} \sum_{\mathbf{k},\mathbf{k}'} \chi_{\mathbf{k}}^* \chi_{\mathbf{k}'} \langle \mathbf{k} | v | \mathbf{k}' \rangle < 0. \quad . \quad . \quad (4.112)$$

This is necessary because the latter expression is less than the former, and sufficient because they differ negligibly for small χ. Finally, the above criterion is equivalent to the existence of a negative eigenvalue ξ of the equation

$$2|\epsilon_k| \chi_{\mathbf{k}} + \sum_{\mathbf{k}'} \langle \mathbf{k} | v | \mathbf{k}' \rangle \chi_{\mathbf{k}'} = \xi \chi_{\mathbf{k}}, \quad . \quad . \quad . \quad . \quad . \quad (4.114)$$

a condition which was obtained by Bogoliubov (1958 b) directly from the condition that the normal solution should be unstable.

This last equation was used by Solovier (1958) who allowed for the possibility of different spin and isobaric spin states being responsible for superfluidity. His equations are only valid for weak potentials, and he appears to have required that there should be a superfluid solution in all angular momentum states in which the interaction is important. Thus he obtained several conditions which have to be satisfied by the potential. In fact, for a sufficiently weak potential, it is necessary and sufficient that $\langle \mathbf{k}_f | v | \mathbf{k}_f \rangle$ should be negative for one particular spin, isobaric spin and angular momentum state in order that (4.76) should have a non-trivial solution (tensor forces prevent these being good eigenstates in general and obviously introduce further complication).

Cooper *et al.* (1959) used (4.112) as the basis of a variational calculation. They considered interaction in s states for which, with

$$f(k) = k\chi_{\mathbf{k}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.115)$$

the appropriate condition is

$$2\int_0^\infty |\epsilon_k| |f(k)|^2 dk + \int_0^\infty v(r) |f(r)|^2 dr < 0 \quad . \quad . \quad (4.116)$$

where

$$f(r) = r\chi(\mathbf{r}) = r(2\pi)^{-3/2} \int \exp(i\mathbf{k}\,.\,\mathbf{r}) \chi_{\mathbf{k}} d\mathbf{k} \quad . \quad . \quad . \quad (4.117)$$

or, from (4.115),

$$f(r) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty f(k) \sin(kr)\, dr. \quad . \quad . \quad . \quad . \quad (4.118)$$

For e_k they used the effective mass approximation; presumably with the value found by Brueckner and Gammel for the effective mass at the fermi surface. With the trial functions they used they were not able to satisfy (4.116). Mills *et al.* (1959) continued this work with improved trial functions. For e_k they considered three possibilities.

(i) An approximate polynomial fit to the Brueckner–Gammel (1958) values (presumably the curve e_k in our fig. 2, though this is not made clear):

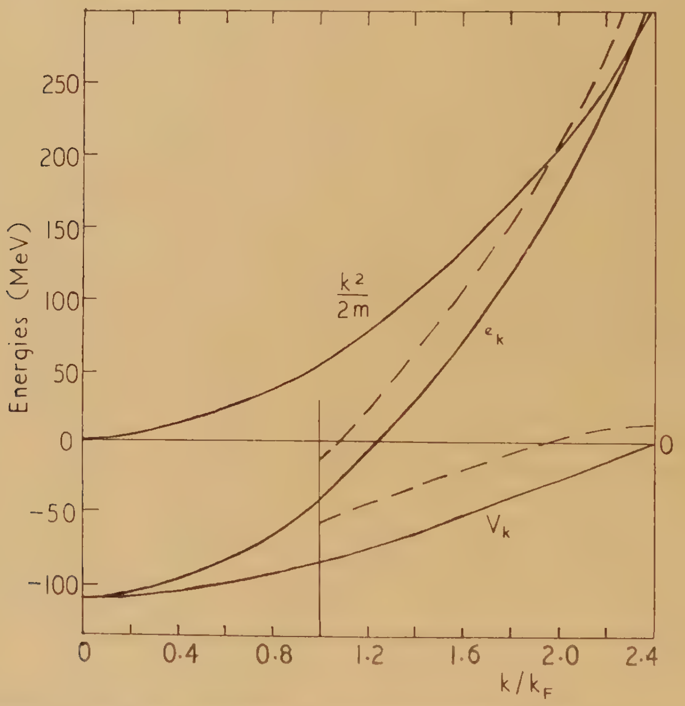
$$e_k = k^2/2M + [-111 + 39{\cdot}1\,(k/k_f)^2 - 8{\cdot}26(k/k_f)^3]\ \text{Mev}, \quad \text{for} \quad k \leqslant 2{\cdot}4\,k_f$$
$$= 0 \quad \text{for} \quad k < 2{\cdot}4\,k_f.$$

(ii)
$$e_k = \frac{k^2}{2M} + \frac{V_0}{1+\alpha k^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.119)$$

with V_0 and α fitted by the requirements that e_{kf} should be the separation energy and the effective mass at the fermi surface should be the Brueckner–Gammel value (which they incorrectly take to be $0{\cdot}67M$), and

(iii)
$$e_k = k^2/2M^*, \quad M^* = 0{\cdot}67\,M,$$
i.e. the effective mass approximation.

Fig. 2



Typical form of the self consistent energy spectrum. For $k > k_f$ the curve labelled e_k is $\bar{e}\,(\mathscr{E}, k)$ defined in (3.129), with $\mathscr{E} \simeq 2ekf$. The broken curve is $\bar{e}(\mathscr{E}, k)$ with $\mathscr{E} = 2e_m$ where $m \approx 0$. These curves are taken from Brueckner and Gammel (1958); they suffer from the error mentioned at the beginning of § IV of that paper.

Equation (4.116) was evaluated in the singlets-state using the Brueckner–Gammel potential given in § 1.3. With $k_f = 1{\cdot}34 f^{-1}$ the inequality was satisfied for all choices of e_k; with $k_f = 1{\cdot}40 f^{-1}$ it was satisfied for the choice

(ii) only, and with $k_f = 1{\cdot}48 f^{-1}$ it was not satisfied for either set of e_k. As pointed out by Mills *et al.* it is possible that better trial wavefunctions would satisfy the inequality in at least some of the other cases. They conclude from this work, therefore, that nuclear matter is superfluid. Their results show, incidentally, that the effective mass approximation grossly overestimates the value of (4.116) compared to the other choices of e_k, and is therefore less likely to give a superfluid state.

Since, as we have seen, the existence of a singularity in the Brueckner K matrix already implies superfluidity on the BCS theory, the calculation of Emery (1959), referred to in §4.2, supports the conclusion that nuclear matter is superfluid. In fact it follows from Emery's work that a superfluid state exists, in the effective mass approximation and at a density corresponding to $k_f = 1{\cdot}5 f^{-1}$, provided $M^* > 0{\cdot}56\,M$. Mills *et al.* do not in fact obtain superfluidity with these parameters, so their trial functions must be inadequate for this case.

The magnitude of the energy gap caused by the singlet s interaction has been estimated by Emery and Sessler (1960). They linearize (4.76) by assuming that, in the region around $k = k_f$ in which $\mu_{\mathbf{k}}$ is not negligible in comparison with ϵ_k, the variation of $\mu_{\mathbf{k}}$ can be ignored, so that it can be replaced by $\mu_{\mathbf{k}_f} = \epsilon_0$ in the demoninator. From (4.96), $2\epsilon_0$ is the required energy gap. Introducing χ by (4.110), we can write for the s-wave part of (4.76) using operator notation.

$$|\chi\rangle = -Gv\,|\chi\rangle \qquad (4.120)$$

where

$$G = \frac{1}{\pi}\int_0^\infty \frac{dk\,|k\rangle\langle k|}{[\epsilon_k{}^2 + \epsilon_0{}^2]^{1/2}} \qquad (4.121)$$

and

$$\langle r\,|\,k\rangle = \sin(kr). \qquad (4.122)$$

Following Emery and Sessler we define

$$|\phi\rangle = |k_f\rangle - G_0 v\,|\phi\rangle \qquad (4.123)$$

with

$$G_0 = \frac{1}{\pi}\int \frac{dk\,[|k\rangle\langle k| - |k_f\rangle\langle k_f|]}{[\epsilon_k{}^2 + \epsilon_0{}^2]^{1/2}}. \qquad (4.124)$$

Then $\langle\phi\,|\,v\,|\,\chi\rangle = \langle k_f\,|\,v\,|\,\chi\rangle\langle\phi\,|\,vG_0v\,|\,\chi\rangle$
from (4.123), and

$$\langle\phi\,|\,v\,|\,\chi\rangle = -\langle\phi\,|\,vGv\,|\,\chi\rangle$$

from (4.120). The difference of the last two equations yields

$$\frac{1}{\langle\phi\,|\,v\,|\,k_f\rangle} = -\frac{1}{\pi}\int_0^\infty \frac{dk}{[\epsilon_k{}^2 + \epsilon_0{}^2]^{1/2}}. \qquad (4.125)$$

Emery and Sessler approximate the integral here by using the effective mass approximation, which should not be too bad since the integrand is strongly peaked about $k = k_f$, and thus obtain

$$2\epsilon_0 = \frac{8\hbar^2 k_f{}^2}{M^*}\exp\left[\frac{\pi}{2M^*}\,\frac{\hbar^2 k_f{}^2}{\langle\phi\,|\,v\,|\,k_f\rangle}\right] \qquad (4.126)$$

In evaluating ϕ from (4.123) they ignore the ϵ_0 in (4.124), which is presumably permissible since the integrand is not peaked around $k = k_f$, and use the form (4.119) for e_k, with e_{k_f} equal to the binding energy per particle and

$$\left.\frac{\partial e(k)}{\partial k}\right|_{k} = \frac{k_f}{M^*}.$$

They plot the energy gap as a function of M^*/M, using the Brueckner–Gammel potential, and with the roughly empirical values $k_f = 1{\cdot}4 f^{-1}$, $e_{k_f} = 15{\cdot}5$ MEV. It is a rapidly increasing function, being about $0{\cdot}1$ MEV for $M^* = M$ and less than $0{\cdot}5 \times 10^{-5}$ MEV for $M^* = 0{\cdot}75\,M$. At a lower density, $k_f = 1{\cdot}0 f^{-1}$, with e_{k_f} taken as 10 MEV, they found $2\epsilon_0 = 3{\cdot}5$ MEV and 1 MEV respectively for the same values of M^*.

An alternative method of calculating the energy gap is proposed in the work of Brueckner, Soda, Anderson and Morel (1960) and Anderson and Morel (1960). To establish this method we write (4.76) in the form

$$\mu = vL\mu \quad \ldots\ldots\ldots \quad (4.127)$$

where

$$L = \sum_{\mathbf{k}} \frac{|\mathbf{k}\rangle\langle\mathbf{k}|}{2[\epsilon_k{}^2 + |\mu_{\mathbf{k}}|^2]^{1/2}}, \quad \ldots\ldots \quad (4.128)$$

and divide L into two parts

$$L = L_1 + L_2 \quad \ldots\ldots\ldots \quad (4.129)$$

where L_1 involves states with $|\epsilon_k| < \delta e$ and L_2 involves states with $|\epsilon_k| > \delta e$ for some (δe) to be chosen later. Then, from (4.127),

$$L_2\mu = L_2 v\,(L_1\mu + L_2\mu)$$

which gives

$$L_2\mu = (1 - L_2 v)^{-1} L_2 v L_1 \mu.$$

On using this to eliminate $L_2\mu$ from (4.127) we obtain

$$\begin{aligned} \mu &= (v + v(1 - L_2 v)^{-1} L_2 v) L_1 \mu \\ &= \tilde{K} L_1 \mu \quad \ldots\ldots\ldots\ldots \quad (4.130) \end{aligned}$$

where

$$\begin{aligned} \tilde{K} &= v + v\,(1 - L_2 v)^{-1} v \\ &= v + vL_2 v + vL_2 vL_2 v + \ldots \quad \ldots\ldots \quad (4.131) \end{aligned}$$

Anderson and Morel propose to choose δe to satisfy (i) $\delta e \gg \mu_{\mathbf{k}}$ so that $\mu_{\mathbf{k}}$ can be ignored in L_2 and (ii) δe sufficiently small for $\langle\mathbf{k}'|\tilde{K}|\mathbf{k}\rangle$ to depend only on the angle between $\mathbf{k}$ and $\mathbf{k}'$ for $|\epsilon_k|$, $|\epsilon_{k'}| < \delta e$. If (ii) is satisfied then, considering for example just the s-state interaction and putting $\langle\mathbf{k}'|\tilde{K}|\mathbf{k}\rangle = -v_0/\Omega$, (4.130) is equivalent to the simple problem discussed in § 4.5 and, for $v_0 > 0$ we have (4.79)

$$2\epsilon_0 = \frac{2\delta e}{\sinh(1/n_f v_0)}, \quad \ldots\ldots\ldots \quad (4.79)$$

For $v_0 < 0$ there is no superfluid solution. Since ϵ_0 is independent of δe, (4.79) requires that v_0 varies rapidly with δe for small δe and in fact tends to infinity as $\delta e \to 0$. The solution of the equation for $\tilde{K}$ does not have this behaviour however. For example, with a separable potential, it is easy to see (cf. the discussion in § 4.2) that $\langle \mathbf{k} | K | \mathbf{k} \rangle$ tends to zero as $k \to k_f$. This illustrates the breakdown here of (4.79) for small δe. Probably it is then permissible neither to take the matrix elements of $\tilde{K}$ constant nor to ignore μ in determining L_2.

Brueckner, Soda, Anderson and Morel (1960) make the further approximation of equating $\tilde{K}$ with the Brueckner K matrix. Apart from the restriction to $|\epsilon_k| > \delta e$ in K, which is to some extent taken account of in K by the method of treating the self-consistency, which introduces a gap in the spectrum (see § 4.2), and by the mesh used in the numerical treatment of the integral equation, these quantities differ by the inclusion of terms with $\epsilon_k < 0$ in $\tilde{K}$. These correspond roughly to hole–hole scattering terms and, although these do not contribute greatly to the total binding energy, they may well be significant in individual K-matrix elements. Thus we do not regard this approximation as reliable. Brueckner and Gammel found $\langle \mathbf{k}_f \mid K \mid \mathbf{k}_f \rangle$ to be positive in the singlet s-state and negative in the triplet s and singlet d-states. Thus, according to this method, the singlet s-state does not lead to superfluidity, a conclusion which is in contradiction with that reached above on the basis of more reasonable assumptions. For the triplet s and singlet d-states, Brueckner *et al.* calculate the gaps to be $0{\cdot}006\,\delta e$ and $10^{-6}\delta e$ respectively. One has little idea of the value of δe which is appropriate to the particular approximation used to calculate K, so it is diffiuclt to give estimates of the energy gap from these figures. Brueckner *et al.* take δe to be the fermi energy (which they take as 57·5 MeV) but this is probably too large for the approximations of constant matrix elements and $K = \tilde{K}$ to be justified.

In conclusion then the existence and magnitude of an energy gap in nuclear matter is seen to be a delicate question. Moreover the gaps which have been found are very small, being at the most about 0·1 MeV. The mean spacing of single particle levels in a cube of volume equal to that of a nucleus with A nucleons is about $(30/A)$ MeV, so that only for the heaviest nuclei would such a gap be significant. Emery and Sessler (1960) suggest that a larger gap might arise in actual nuclei (see Bohr *et al.* 1958 for a discussion of the empirical facts) because of the lower mean density due to the large surface region. Such ideas cannot, however, be taken very seriously because the 'correlation length' (Bardeen *et al.* 1957) is so large. In fact the range of momenta involved in the BCS phenomena is of the order of $M\epsilon_0/k_f$ so that the correlation length, which is the inverse of this, is

$$\frac{k_f^2}{2M\epsilon_0}\frac{2}{k_f} \approx \frac{60}{\epsilon^0} f$$

where ϵ_0 is in MeV. For all plausible values of ϵ_0 this is larger than any nucleus. It is not surprising therefore that the pairing effect in nuclei should depend greatly on detailed properties of the particular nucleus considered (Beliaev 1958).

In the volume energy of nuclear matter the BCS contribution per particle is much smaller than the gap ϵ_0, since only a small number of particles near to the fermi surface are involved in the latter. It is unlikely that extrapolation from actual nuclei, with subtraction of surface effects, shell effects, etc., will ever fix the binding energy of nuclear matter with such a precision that such a contribution becomes significant.

§ 5. CONCLUSION

The quantitive success of the theory cannot be accurately judged at present. On the one hand, the empirical properties of nuclear matter are ill-defined (§ 1.1). This applies even to the binding energy per particle, which varies by several MEV between different versions of the mass formula (Green 1958, Cameron 1957). On the other hand, although this is not clear from the literature†, the Brueckner–Gammel calculation is sensitive to details of the internucleon force not well determined by the scattering experiments. In particular the experimental uncertainty in the triplet central odd state interaction permits several MEV variation in the calculated energy per particle. Thus the numerical results could not, for example, rule out the possibility of substantial contributions from many body forces‡. In forming a conclusion we can only consider whether there exists a theory which, were the two body force accurately known, would allow a reliable evaluation of the consequences.

The Jastrow approach is inadequate here. It does not run into the singularity troubles of Brueckner theory, and we think it would give sensible results if worked out with realistic forces. But these results would not be accurate, for the cluster correlations appear to be large. There are also other corrections, because of the approximate nature of the basic Jastrow wavefunction, which have not been estimated.

The singularity in Brueckner theory is easily lost in practice, either by approximations in the working or, as with Moszkowski and Scott (§ 3.5), by not iterating the attractive tail of the potential. The lowest order corrections then appear to be small, of order 1 MEV in the binding energy per particle at normal density. However, the methods of estimating these (§ 3.6) are very crude. They give, for instance, a much smaller self-consistency correction then that finally obtained by Brueckner and Gammel. These estimates should be repeated more realistically, including in particular the tensor force.

Although the quantitative effect should be very small, the Brueckner method will be formally satisfactory only when the singularity is eliminated rather than lost. The correct description of the fermi surface is probably that of Bardeen, Cooper, and Schrieffer, although their theory is convincingly established only for a grossly non-local ('reduced') hamiltonian. This account of the long range correlations should be combined with the Brueckner account of the short range. This may come about by, for example, a coherent formulation of the separate treatment of low and high energy virtual excitations (Brueckner *et al.* 1960), or by a practical version of the Green function scheme.

---

† See for example the discussion following the report by Brueckner to the 1960 Kingston conference (Brueckner 1960).

‡ See, in this connection, Derrick *et al.* (1961).

ACKNOWLEDGEMENTS

We are indebted to many people over a long period for valuable discussion of this subject. Much of the work involved in writing this paper was performed while we were at the Atomic Energy Research Establishment, Harwell. One of us (E. J. S.) acknowledges the assistance of the I.C.I. Fellowship Fund at Cambridge.

REFERENCES

ANDERSON, P. W., 1958, *Phys. Rev.*, **110,** 827.
ANDERSON, P. W., and MOREL, P., 1960, *Phys. Rev. Letters*, **5,** 136.
AVILES, J. B., 1958, *Annals of Physics*, **5,** 251.
BARDEEN, J., COOPER, L. H., and SCHRIEFFER, J. R., 1957, *Phys. Rev.*, **108,** 1175.
BAYMAN, B. F., 1959, *Nucl. Phys.*, **15,** 33.
BELIAEV, S. T., 1958, *The Many Body Problems* (Les Houches, 1958) (New York: J. Wiley), p. 377.
BELL, J. S., 1959, *Proc. phys. Soc., Lond.*, **73,** 118; 1960, *Proc. of Int. Spring School of Physics, Naples* (to be published by Acad. Sci. Press).
BETH, E., and UHLENBECK, G. E., 1936, *Physica*, **3,** 729.
BETHE, H. A., 1956, *Phys. Rev.*, **103,** 1353.
BETHE, H. A., and GOLDSTONE, J., 1957, *Proc. roy. Soc.* A, **238,** 551.
BLATT, J. M., and WEISSKOPF, V. F., 1952, *Theoretical Nuclear Physics* (New York: J. Wiley).
BLATT, J. M., and DERRICK, G., 1958, *Nucl. Phys.*, **3,** 310.
BLOCH, C., and DE DOMINICIS, C., 1958, *Nucl. Phys.*, **7,** 459.
BLOCH, C., 1960 a, *Proc. Int. Conf. on Nuclear Structure* (Kingston) (Amsterdam: North Holland; Toronto: University of Toronto Press), p. 7; 1960 b, Proc. Int. Cong. on Many Particle Problems (Utrecht), *Physica*, **26,** (Suppl.), 62.
BODMER, A. R., 1958, *Nucl. Phys.*, **9,** 371.
BOGOLIUBOV, N. N., 1947, *J. Phys.*, (*U.S.S.R.*), **11,** 23; 1958 a, *J. exp. theor. Phys.* (*U.S.S.R.*), **11,** 58 and 73 (English translation in *J.E.T.P.*, **34** **(7),** 41 and 51); 1958 b, *Dokl. Acad. Nauk SSSR*, **119,** 52 (English translation in *Sov. Phys. Dokl.*, **4,** 279); 1960, Proc. of Int. Cong. on Many Particle Problems (Utrecht), *Physica*, **26,** (Suppl.), 1.
BOGOLIUBOV, N. N., TOLMACHEV, V. V., and SHIRKOV, D. V., 1959, *A new method in the theory of superconductivity* (New York: Consultants Bureau).
BOGOLIUBOV, N. N., ZUBAREV, D. N., and TSERKOVNIKOV, YU. A., 1958, *Dokl. Acad. Nauk SSSR*, **117,** 788 (English translation in *Sov. Phys. Dokl.*, **2,** 535).
BOHR, A., MOTTELSON, B. R., and PINES, D., 1958, *Phys. Rev.*, **110,** 936.
BREIT, G., 1958, *Rev. mod. Phys.*, **30,** 507.
BRENIG, W., 1957, *Nucl. Phys.*, **4,** 363.
BROUT, R., 1960, *Phys. Rev. Letters*, **5,** 193.

Brueckner, K. A., 1954, *Phys. Rev.*, **96,** 508; 1955 a, *Ibid.*, **97,** 1353; 1955 b, *Ibid.*, **100,** 36; 1958, *The Many Body Problem* (Les Houches, 1958) (New York: J. Wiley), p. 47; 1960, *Proc. Int. Conf. on Nuclear Structure* (Kingston) (Amsterdam: North Holland; Toronto: University of Toronto Press), p. 67.

Bruekcner, K. A., Eden, R. J., and Francis, N. C., 1955, *Phys. Rev.*, **98,** 1445.

Brueckner, K. A., and Gammel, J. L., 1957, *Phys. Rev.*, **105,** 1679; 1958, *Ibid.*, **109,** 1023.

Brueckner, K. A., Gammel, J. L., and Kubis, J. T., 1960, *Phys. Rev.*, **118,** 1095.

Brueckner, K. A., and Goldman, D. T., 1960, *Phys. Rev.*, **117,** 207.

Brueckner, K. A., Levinson, C. A., and Mahmoud, H. M., 1954, *Phys. Rev.*, **95,** 217.

Brueckner, K. A., and Levinson, C. A., 1955, *Phys. Rev.*, **97,** 1344.

Brueckner, K. A., Soda, T., Anderson, P. W., and Morel, P., 1960, *Phys. Rev.*, **118,** 1442.

Brueckner, K. A., and Wada, W., 1956, *Phys. Rev.*, **103,** 1008.

Cameron, A. G. W., 1957, *Canad. J. Phys.*, **35,** 1021.

Chisholm, J. S. R., and Squires, E. J., 1959, *Nucl. Phys.*, **13,** 156.

Clark, J. W., and Feenberg, E., 1959, *Phys. Rev.*, **113,** 388.

Coester, F., 1958, *Nucl. Phys.*, **7,** 421.

Coester, F., and Kümmel, H., 1960, *Nucl. Phys.*, **17,** 477.

Cooper, L. N., 1956, *Phys. Rev.*, **104,** 1189.

Cooper, L. N., Mills, R. L., and Sessler, A. M., 1959, *Phys. Rev.*, **114,** 1377.

Dabrowski, J., 1958 a, *Proc. phys. Soc., Lond.*, **71,** 658; 1958 b, *Ibid.*, **72,** 499; 1958 c, *Bulletin de l'Academie Polonaise des Sciences, Series des Sci. Math. Astr. et Phys.*, **6,** 635.

Derrick, G., Mustard, D., and Blatt, J. M., 1961, *Phys. Rev. Letters*, **6,** 69.

Dingle, R. B., 1949, *Phil. Mag.*, **40,** 573.

de Dominicis, C., and Martin, P. C., 1957 a, *Phys. Rev.*, **105,** 1417; 1957 b, *Ibid.*, **105,** 1419; 1958, *Bull. Amer. phys. Soc.*, **3,** 224.

Drell, S., and Huang, K., 1953, *Phys. Rev.*, **91,** 1527.

Eden, R. J., and Francis, N. C., 1955, *Phys. Rev.*, **97,** 1366.

Eden, R. J., 1956, *Proc. roy. Soc.* A, **235,** 408; 1958, *Nucl. Phys.*, **9,** 65; 1959, *Nuclear Reactions*, Vol. I, Ch. 1 (Amsterdam: North Holland Publishing Co.).

Elton, L. R. B., 1958, *Nucl. Phys.*, **5,** 173; 1961, *Nuclear Sizes* (to be published)

Emery, V. J., 1958, *Nucl. Phys.*, **6,** 585; 1959, *Ibid.*, **12,** 69; 1960, *Ibid.*, **19,** 154.

Emery, V. J., and Sessler, A. M., 1960, *Phys. Rev.*, **119,** 248.

Euler, H., 1937, *Z. Phys.*, **105,** 553.

Fano, G., and Tomasini, A., 1960, *Nuovo Cim.*, **18,** 1247.

Ford, K. W., and Hill, D. L., 1955, *Ann. Nucl. Sci.*, **5,** 46.

Fujita, J., and Mujazawa, H., 1957, *Progr. theor. Phys.*, **17,** 360.

Fukuda, N., and Newton, R., 1956, *Phys. Rev.*, **103,** 1558.

Galitskii, V. M., and Migdal, A. B., 1958, *J. exp. theor. Phys.*, **34,** 139 (English translation in *J.E.T.P.*, **34,** **(7),** 96).

Galitskii, V. M., 1958, *J. exp. theor. Phys.*, **34,** 151 (English translation in *J.E.T.P.*, **34** **(7),** 104).

Gammel, J. L., and Thaler, R. M., 1960, *Progress in Elementary Particle and Cosmic Ray Physics*, **5,** 99.

Gaudin, M., 1960 a, *Nucl. Phys.*, **15,** 89; 1960 b, *Ibid.*, **20,** 520.

Goldstone, J., 1957, *Proc. roy. Soc.* A, **239,** 267.

Gomes, L. C., Walecka, J. D., and Weisskopf, V. F., 1958, *Annals of Physics*, **3,** 241.

Gorkov, L., 1958, *Zhurnal Exsp. I Teor. Fiz.*, **34**, 735 (English translation in *J.E.T.P.*, (7), 505).

Gottfreid, K., 1958, unpublished CERN report.

Green, A. E. S., 1958, *Rev. mod. Phys.*, **30**, 569.

Hamilton, J., 1959, *The Theory of Elementary Particles* (Oxford).

Hartogh, C. D., and Tolhoek, H. A., 1958, *Physica*, **24**, 721, 875 and 896.

Henley, E. M., and Willets, L., 1961, preprint (CERN 1408).

Huang, K., and Yang, C. N., 1957, *Phys. Rev.*, **105**, 767.

Huby, R., 1949, *Proc. phys. Soc., Lond.*, A, **62**, 62.

Hugenholtz, N. M., 1957 a, *Physica*, **23**, 481; 1957 b, *Ibid.*, **23**, 533.

Hugenholtz, N. M., and Van Hove, L., 1958, *Physica*, **24**, 363.

Iwamoto, F., and Yamada, Y., 1957 a, *Progr. theor. Physics.*, **17**, 543; *Ibid.*, 1957 b, *Ibid.*, **18**, 345.

Iwamoto, G., 1958, *Progr. theor. Phys.*, **19**, 597; 1959, *Ibid.*, **22**, 903; 1960, *Ibid.*, **23**, 871.

Jastrow, R., 1955, *Phys. Rev.*, **98**, 1479.

Jost, R., and Pais, A., 1951, *Phys. Rev.*, **82**, 840.

Katz, A., 1960 a, *Nucl. Phys.*, **18**, 177; 1960 b, *Ibid.*, **20**, 663.

Katz, A., de Shalit, A., and Talmi, I., 1960, *Nuovo Cim.*, **16**, 485.

Khalatnikov, I. M., and Abrikosov, A. A., 1959, *Advanc. Phys.*, **8**, 45.

Klein, A., and Prange, R. E., 1958, *Phys. Rev.*, **112**, 994.

Klein, A., 1961, *Phys. Rev.* (to be published).

Kohn, W., and Luttinger, J. M., 1960, *Phys. Rev.*, **118**, 41.

Kohn, W., and Nettel, S. J., 1960, *Phys. Rev. Letters*, **5**, 8.

Kümmel, H., 1957, *Nuovo Cim.*, **6**, 23; 1961, *Nucl. Phys.*, **22**, 177.

Kuper, C. G., 1959, *Advanc. Phys.*, **8**, 1.

Leigh, R. S., 1958, unpublished.

Lenz, W., 1929, *Z. Phys.*, **56**, 778.

Levinger, J. S., Razavy, M., Rojo, O., and Webre, N., 1960 a, *Phys. Rev.*, **119**, 130.

Levinger, J. S., 1960, *Nucl. Phys.*, **19**, 370.

Luttinger, J. M., and Ward, J. C., 1960, *Phys. Rev.*, **118**, 1417.

MacGregor, M. H., Moravcsik, M. J., and Stapp, H. P., 1960, *Annual Reviews of Nuclear Science*, **10**, 291.

Martin, P. C., and Schwinger, J., 1959, *Phys. Rev.*, **115**, 1342.

Martin, P. C., and Puff, R. D., 1960, *Bull. Amer. phys. Soc.*, **5**, 30,

Martin, P. C., and Kadanoff, L. P., 1961, to be published.

Matsubara, T., 1955, *Progr. theor. Phys.*, **14**, 351.

Mehta, M. L., 1959, *Nucl. Phys.*, **10**, 333; 1960, preprint.

Mills, R. L., Sessler, A. M., Moszkowski, S. A., and Shankland, D. G., 1959, *Phys. Rev. Letters*, **3**, 381.

Mittelstaedt, P., 1958, *Nucl. Phys.*, **9**, 116.

Moszkowski, S. A., 1959, unpublished notes of lecture at U.C.L.A.

Moszkowski, S. A., and Scott, B. L., 1960, *Annals of Physics*, **11**, 65.

Moszkowski, S. A., and Sessler, A. M., 1960, *Nucl. Phys.*, **18**, 669.

Mottelson, B. R., 1958, *The Many-Body Problem* (Les Houches, 1958) (New York: J. Wiley), p. 283.

Nambu, Y., 1960, *Phys. Rev.*, **117**, 648.

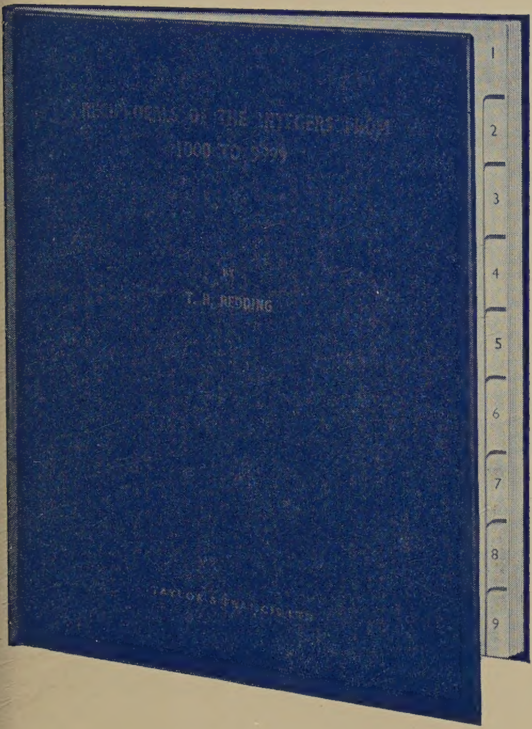
Overhauser, A. W., 1960, *Phys. Rev. Letters*, **4**, 415.

Peierls, R. E., 1938, *Phys. Rev.*, **54**, 918; 1960, *Proc. Int. Conf. on Nuclear Structure* (Kingston), (Amsterdam: North Holland; Toronto: University of Toronto Press), p. 7.

Phillips, R. J. N., 1959, *Rep. Progr. Phys.*, **22**, 562.

Prange, R. E., and Klein, A., 1958, *Phys. Rev.*, **112**, 1008.

Prange, R. E., 1960, *Proc. of Int. Spring School of Physics, Naples* (to be published by Acad. Sci. Press).
Ravenhall, D. G., 1958, *Rev. mod. Phys.*, **30,** 430.
Riesenfeld, W., and Watson, K. M., 1956, *Phys. Rev.*, **104,** 492.
Rickaysen, G., 1959, *Phys. Rev.*, **115,** 795.
Rodberg, L. S., 1957, *Annals of Physics*, **2,** 199.
Salam, A., 1953, *Progr. theor. Phys.*, **9,** 553.
Sawada, K., and Fukuda, N., 1961, preprint (CERN, 1408).
Schiff, L. I., 1952, *Phys. Rev.*, **86,** 856; 1954, *Proc. of Int. Conf. on Theor. Physics, Tokio.*
Skyrme, T. H. R., 1955, *Proc. roy. Soc.* A, **226,** 521; 1956, *Phil. Mag.*, **1,** 1043.
Smith, R. C., and Sharp, R. J., 1960, *Canad. J. Phys.*, **9,** 1154.
Soloviev, V. G., 1958, *Nuovo Cim.*, **10,** 1022.
Squires, E. J., 1958, *Nucl. Phys.*, **8,** 41.
Swiatecki, W. J., 1956, *Phys. Rev.*, **103,** 265.
Tagami, T., 1959, *Progr. theor. Phys.*, **21,** 533.
Temkin, A., 1960, *Annals of Physics*, **9,** 93.
Thouless, D. J., 1957, *Phys. Rev.*, **107,** 559; 1958, *Ibid.*, **112,** 906; 1959 a, *Ibid.*, **114.** 1383; 1959 b, *Ibid.*, **116,** 21; 1960 a, *Ibid.*, **117,** 1256; 1960 b, *Annals of Physics*, **10,** 553.
Trainor, L. E. H., Bhatia, A. B., and Sood, P. C., 1958, *Nucl. Phys.*, **9,** 287.
Trainor, L. E. H., and Sood, P. C., 1959, *Nucl. Phys.*, **12,** 629.
Tobocman, W., 1957, *Phys. Rev.*, **107,** 203.
Valatin, J. G., 1958, *Nuovo Cim.*, **7,** 843.
Wada, Y., Takano, F., and Fukuda, N., 1958, *Progr. theor. Phys.*, **19,** 597.
Watson, K. M., 1956, *Phys. Rev.*, **103,** 489.
Weisskopf, V. F., 1950, *Helv. phys. acta*, **23,** 187; 1957, *Nucl. Phys.*, **3,** 423.
Weisskopf, V. F., and de Shalit, A., 1958, *Annals of Physics*, **5,** 282.
Wentzel, G., 1960, *Helv. phys. Acta.*, **33,** 859.
Werner, E., 1959, *Nucl. Phys.*, **10,** 688.
Wigner, E. P., 1936, *Proc. nat. Acad., Sci., Wash.*, **22,** 662.
Wild, W., 1960, *Z. Phys.*, **158,** 322.
Willets, L., 1958, *Rev. mod. Phys.*, **30,** 542.
de Witt, B., 1956, *Phys. Rev.*, **103,** 1565.
Yamazaki, K., 1961, *Nucl. Phys.*, (to be published).
Yamaguchi, Y., and Yamaguchi, Y., 1954, *Phys. Rev.*, **95,** 1635.
Van Hove, L., 1959, *Physica*, **25,** 498; 1960, *Proc. Int. Cong. on Many Body Problems (Utrecht), Physica*, **26,** Suppl., 200.



FOR USERS OF CALCULATORS

RECIPROCALS OF THE INTEGERS FROM 1000 TO 9999

By T. H. Redding
M.Sc.(Lond.), A.M.I.Mech.E., A.F.R.Ae.S.

Arranged for use with mechanical calculating machines to facilitate the evaluation of quotients and compound fractions

With an Appendix on mechanical barrel-setting calculators

These tables list the 10^6 multiples of the reciprocals of the integers 1000 to 9999 and are displayed as 1000 entries on each of nine double-page spreads which are thumb-indexed (1000 . . . , 2000 . . . , 3000 . . . , . . . etc) to facilitate rapid manipulation with one hand only. The entries are direct reading and are correct to the nearest integer in the third place of decimals.

Although suitable for general use, the tables have been prepared and arranged in a manner particularly suitable for use in conjunction with mechanical computing machines of the "barrel-setting" type in the evaluation of quotients and compound fractions. Equally the tables find application in effecting the evaluation of quotients on adding and listing machines when these incorporate a mechanism for automatically evaluating simple products.

In the first instance it was anticipated that an Appendix dealing with the advantages to be gained by the use of the reciprocal function in conjunction with such machines—and particularly with barrel-setting machines—would suffice to ensure the best use being made of the tables. The present 22-page *Appendix*, however, goes further than this, since the discussion of the tables is prefaced by a description of barrel-setting machines and their (simplest) method of operation in the evaluation of products, quotients and compound fractions. Thus, the description of each calculating procedure is followed by a numerical example with accompanying diagrams to illustrate the keyboard-displays at the beginning, the end, and at intermediate stages in the calculation. The machine-evaluation of series comprising the sum of a number of product or quotient-terms is also discussed. Generally it is hoped that the Appendix will serve as an introduction to mechanical methods of computation and that it will materially assist prospective purchasers in the choice of a machine.

A valuable guide to all users of Computers. Bound in blue publishers case, lettered on the front, with an 18 page table index. Printed on good quality paper for constant use.

Size 10 in. × 8 in. *Price* 18s. 6d. *plus postage and packing* 1s. 3d.

Printed and Published by

TAYLOR & FRANCIS LTD

RED LION COURT, FLEET STREET, LONDON, E.C.4

FILED IN STACKS

Announcing the new publication

INSTRUMENT CONSTRUCTION

Translated from the Russian

SIRA

Приборостроение

Editor-in-chief: M. E. Rakovskii. Sub-editor: Yu. I. Shendler

The Russians describe their publication as a 'scientific, technical and production' journal. It covers industrial instruments and instrumentation, automatic control, and production engineering for precision work. The articles which it presents not only introduce new instruments and techniques, they also afford a valuable insight into current Russian practice.

Subscription £6 yearly post free ($17.10 U.S.A. and Canada). A special rate of £3 yearly post free ($8.55 U.S.A. and Canada) is available to University and Technical College libraries.

Single copies 15*s*. each ($2.15 U.S.A. and Canada) plus postage

Orders should be sent to the Subscription Department, Taylor & Francis Ltd.

Produced for the Department of Scientific and Industrial Research by the
British Scientific Instrument Research Association,
'Sira', South Hill, Chislehurst, Kent

Printed and Published for B.S.I.R.A. by
TAYLOR & FRANCIS LTD
RED LION COURT, FLEET STREET, LONDON, E.C.4